Physics 513, Quantum Field Theory

Homework 1

Due Tuesday, 9th September 2003

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Problem 1) The conservation of four-momentum implies that in particle one's rest frame,

$$p_1^0 = m_1 = E_2 + E_3. \tag{1.1}$$

By the invariance of p_1^2, p_2^2, and p_3^2, it is clear that,

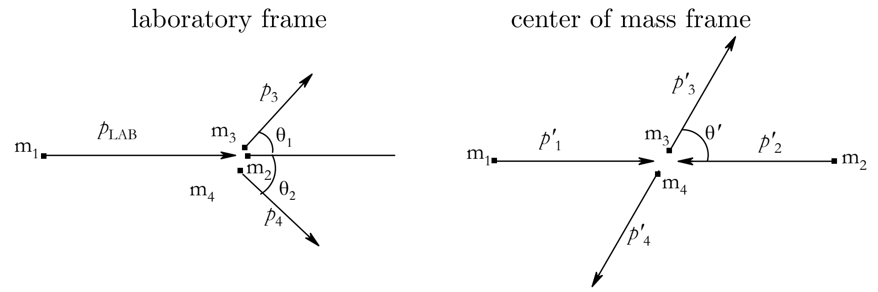
$$\begin{aligned} p_1^2 = m_1^2 &= (p_2 + p_3)^2, \\ &= p_2^2 + p_3^2 + 2p_2p_3, \\ &= m2^2 + m_3^2 + 2E_2E_3 - \vec{p}_2\vec{p}_3. \end{aligned}$$

But in particle one's rest frame, $\vec{p}_2 = -\vec{p}_3$ and by (1.1), $E_3 = m_1 - E_2$. Therefore,

$$\begin{aligned} m_1^2 &= m_2^2 + m_3^2 + 2m_1E_2 - 2\left(E_2^2 - \vec{p}_2^2\right), \\ &= m_3^2 - m_2^2 + 2m_1E_2, \\ \therefore E_2 &= \frac{m_1}{2} + \frac{m_2^2 - m_3^2}{2m_1}. \end{aligned} \tag{1.2}$$

Problem 2)

(a)



(b) In the center of mass frame of reference, the total 4-momentum can be described by,

$$p_{cm} = p_1' + p_2' = (E_1 + E_2; \vec{0}) \equiv (E_{cm}; \vec{0}).$$

Note that p_1p_2 is an invariant scalar product. Evaluated in the laboratory frame,

$$p_1p_2 = E_Lm_2 - \vec{p}_L\vec{0} = E_Lm_2.$$

This allows us to conclude that,

$$\begin{aligned} p_{cm}^2 = E_{cm}^2 &= p_1'^2 + p_2'^2 + 2p_1p_2, \\ \therefore E_{cm}^2 &= m_1^2 + m_2^2 + 2E_Lm_2. \end{aligned} \tag{2.1}$$

(c) Consider the four-vectors η and λ defined by,

$$\eta \equiv (p_1 + p_2) = (E_L + m_2; \vec{p}_L) \qquad \eta' \equiv (E_1' + E_2'; \vec{0}) = (E_{cm}; \vec{0});$$
$$\lambda \equiv (p_1 - p_2) = (E_L - m_2; \vec{p}_L) \qquad \lambda' \equiv (E_1' - E_2'; 2\vec{p}\,').$$

By the frame-invariance of the scalar product,

$$\eta\lambda = \eta'\lambda' = E_L^2 - m_2^2 - |\vec{p}_L|^2 = E_{cm}(E_1' - E_2'). \tag{2.2}$$

Now consider the identity $\eta'^2\lambda'^2 = \eta^2\lambda^2$. Calculating these products and using the result above,

$$\eta'^2\lambda'^2 = E_{cm}^2\left((E_1' - E_2')^2 - 4|\vec{p}_1'|^2\right) = \left((E_L + m_2)^2 - |\vec{p}_L|^2\right)\left((E_L - m_2)^2 - |\vec{p}_L|^2\right) = \eta^2\lambda^2,$$

$$E_{cm}^2(E_1' - E_2')^2 - 4|\vec{p}_1'|^2E_{cm}^2 = \left(E_L^2 - m_2^2 - |\vec{p}_L|^2\right)^2 - 4m_2^2|\vec{p}_L|^2,$$

$$= E_{cm}^2(E_1' - E_2')^2 - 4m_2^2|\vec{p}_L|^2,$$

$$\therefore |\vec{p}_1'|^2 = \frac{m_2^2|\vec{p}_L|^2}{E_{cm}^2} \Rightarrow |\vec{p}_1'| = \frac{m_2|\vec{p}_L|}{E_{cm}}. \tag{2.3}$$

(d) By the conservation of four-momentum, $q = p_1 - p_3 = p_4 - p_2$. So,

$$q^2 = (p_4 - p_2)^2 = 2m_4^2 - 2p_2p_4,$$

$$= 2m_4^2 - 2E_4m_4,$$

$$\therefore q^2 = -2m_4(E_4 - m_4). \tag{2.4}$$

(e) The first part of this problem, namely that $s \equiv (p_1 + p_2)^2 = E_{cm}^2$, was demonstrated and used in part (b) above. Let us now consider $t \equiv q^2$,

$$t \equiv q^2 = p_1^2 + p_3^2 - 2p_1p_3 = 2m_1^2 - 2E_1'E_3' + 2|\vec{p}_1\ '||\vec{p}_3\ '|\cos(\theta').$$

Here, we wrote p_1p_2 explicitly in the center of mass frame. Because it is an invariant, any frame will do. Now we can use the fact that $m_1 = m_3$ and $m_2 = m_4$ to see that $|\vec{p}_1'| = |\vec{p}_3'|$ and that $E_1' = E_3'$ by using part (c) from above. We will now use the notation of the assignment where $|\vec{p}_1'| = p'$. This quickly reduces the above equality to

$$q^2 = 2\left(m_1^2 - E_1'^2 + p'^2\cos(\theta')\right).$$

This can be simplified in two ways. First, notice that $m_1^2 - E_1'^2 = -p'^2$ because $E_1'^2 - p'^2 = m_1^2$. Second we will use the trigonometric identity $1 - \cos(\alpha) = 2\sin^2(\alpha/2)$. Introducing these simplifications we obtain

$$q^2 = -4p'^2\sin^2(\theta'/2). \tag{2.5}$$

(f) To explore new areas of physics at very high energies, one requires the greatest center of mass energy possible. This is because the center of mass energy is what is available to create new matter in a collision. It is simple to show that fixed-target experiments have significantly lower energy than comparable colliders. This is seen by solving the expression for s in part (e) above. In a fixed target collision, we can compute $(p_1+p_2)^2$ in the laboratory frame because it is an invariant. In the laboratory frame, $p_1 = (E_B; \vec{p}_L)$ and $p_2 = (m_2, \vec{0})$. Therefore in a fixed target experiment,

$$E_{cm}^2 = p_1^2 + p_2^2 + 2p_1p_2 = m_1^2 + m_2^2 + 2m_2E_B. \tag{2.6}$$

Approximating this in the case of a high energy collision where $E_B >> m_1, m_2$,

$$E_{cm} \simeq \sqrt{2m_2E_B}. \tag{2.7}$$

This does not look very cost effective. If you increased the beam energy 100 times, there would only be 10 times more energy available for particle creation. In the center of mass collision, however, we see that there is much higher efficiency. In such a collision, $p_1 = (E_B; \vec{p})$ and $p_2 = (E_B; -\vec{p})$. Taking the same approximation that the beam energy is significantly higher than the rest-masses of the particles involved,

$$E_{cm} \simeq 2E_B. \tag{2.8}$$

It is clear that this would be the preferred experiment. A 100 fold increase in beam energy would result in 100 times more energy available: the way one would expect it to be. Despite the energy efficiency of center of mass colliders, many experiments still use fixed target experiments. Why? There are several primary reasons. The first is that it is extraordinarily difficult and usually very expensive to build a collider. If the collider is to work with matter and antimatter like Fermilab today, LEP I or LEP II, one can use the same (vertical) magnetic fields to accelerate the particles and antiparticles in opposite directions. This saves money on magnets but requires solving enormous engineering obstacles. In the LEP accelerator at CERN, for example, both the e^- and e^+ beams were in the same vacuum chamber; they had to be prevented from interacting except in very explicit locations along the accelerator. Imagine ultra-relativistic beams of positrons and electrons moving oppositely in a small vacuum tube only separated by a centimeter. It clearly takes a great deal of forethought.

In addition to engineering hurdles, there are also very large costs involved in building these accelerators. If the collider is built to accelerate only matter, then the same magnetic field cannot be used to accelerate opposing beams. This means that literally two entire magnetic tracks must be built (essentially two entirely separate accelerators). This is what is being done for the Large Hadron Collider at CERN.

Problem 3) We would like to consider the Lagrangian density,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - a\phi - \frac{b}{2}\phi^2 - \frac{\alpha}{3!}\phi^3 - \frac{\beta}{4!}\phi^4,$$

under the transformation $\phi \to \phi' = \phi + c$. By direct calculation,

$$\begin{aligned}\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 &- c\left(a + \frac{bc}{2} + \frac{\alpha c^2}{6} + \frac{\beta c^3}{24}\right)\\ &- \phi\left(a + bc + \frac{\alpha c^2}{2} + \frac{\beta c^3}{6}\right)\\ &- \phi^2\left(\frac{b}{2} + \frac{\alpha c}{2} + \frac{\beta c^2}{4}\right)\\ &- \phi^3\left(\frac{\alpha}{6} + \frac{\beta c}{6}\right)\\ &- \phi^4\frac{\beta}{4!}.\end{aligned}$$

We are to show that a constant c can be chosen to remove the linear term in the Lagrangian. Notice that the constant term in the Lagrangian is fine—we can always shift the Lagrangian density by a constant without changing the equations of motion. Therefore, we must show that we can find a c such that,

$$\left(a + bc + \frac{\alpha c^2}{2} + \frac{\beta c^3}{6}\right) = 0; \tag{3.1}$$

Although it would be a terrible headache to solve the above cubic equation in complete generality (short of citing Cardan's solution), we will simply note that every third order polynomial has one real root. Analytically, one sees that for $c \to -\infty$, the expression in parenthesis will eventually be negative and for $c \to \infty$, the expression will eventually be positive. Therefore, thre must be some c such that the above expression vanishes.

After a bit of algebra, one sees that one can shift $\mathcal{L}$ to the form,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{m^2}{2}\phi^2 - \frac{g}{3!}\phi^3 - \frac{\lambda}{4!}\phi^4, \tag{3.2}$$

where,

$$\begin{aligned}\lambda &= \beta;\\ g &= (\alpha - \beta c);\\ m^2 &= \left(b - \alpha c + \frac{\beta c^2}{2}\right);\\ c &= \frac{-4\alpha \pm 2\sqrt{4\alpha^2 - 9\beta b}}{3\beta}.\end{aligned}$$

Physics 513, Quantum Field Theory

Homework 2

Due Tuesday, 16th September 2003

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1. a) Studying classical field theory, we derived the Euler-Lagrange equations of motion,

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0.$$

It is trivial to show that a field which is described by the Lagrangian given has the following equation of motion:

$$-m^2\phi - \frac{\partial V}{\partial \phi} - \partial_\mu \partial^\mu \phi = 0,$$

$$\Longrightarrow \left(\partial_\mu \partial^\mu + m^2\right)\phi = -\frac{\partial V}{\partial \phi}. \tag{1.1}$$

Which is precisely the Klein-Gordon equation for a field in a potential V.

b) The canonical momentum is,

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \partial_0 \phi. \tag{1.2}$$

Using π, we write the Hamiltonian for the field.

$$\begin{aligned} H = \int d^3x \mathcal{H} &= \int d^3x \, (\pi \partial_0 \phi - \mathcal{L}), \\ &= \int d^3x \left(\pi^2 - 1/2(\partial_0\phi)^2 + 1/2(\nabla\phi)^2 + 1/2 m^2\phi^2 + V(\phi)\right), \\ &= \frac{1}{2}\int d^3x \left(\pi^2 + (\nabla\phi)^2 + m^2\phi^2 + 2V(\phi)\right). \end{aligned} \tag{1.3}$$

c) With a complex scalar field, the Lagrangian becomes

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi - V(\phi^*\phi).$$

Following the same procedure as in part (a) above, we use the Euler-Lagrange equation to show that

$$-m^2\phi^*\phi - \phi^*\frac{\partial V}{\partial \phi} - \phi\frac{\partial V}{\partial \phi^*} - \partial_\mu\phi^*\partial^\mu\phi = 0.$$

$$\Longrightarrow \left(\partial_\mu \partial^\mu + m^2\right)\phi^*\phi = -\phi^*\frac{\partial V}{\partial \phi} - \phi\frac{\partial V}{\partial \phi^*} \tag{1.4}$$

It is relatively easy to show that canonical momenta of the field are

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \partial_0 \phi^*;$$

$$\pi^* = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi^*)} = \partial_0 \phi.$$

Using this expression for π, we will proceed as above to compute the Hamiltonian.

$$\begin{aligned} H = \int d^3x \mathcal{H} &= \int d^3x \, (\pi \partial_0 \phi - \mathcal{L}), \\ &= \int d^3x \left(\pi^*\pi - 1/2\pi^*\pi + 1/2\nabla\phi^*\nabla\phi + 1/2m^2\phi^*\phi + V(\phi^*\phi)\right), \\ &= \frac{1}{2}\int d^3x \left(\pi^*\pi + \nabla\phi^*\nabla\phi + m^2\phi^*\phi + 2V(\phi^*\phi)\right). \end{aligned} \tag{1.5}$$

d) Let us derive the Noether current generated by a *global* phase rotation $\phi \to \phi' = e^{i\alpha}\phi$. It is clear that $\mathcal{L}' = \mathcal{L}$ because only modulus terms of ϕ appear in $\mathcal{L}$. We rewrite the global phase rotation to the first order as

$$\begin{aligned} \phi \to \phi' = e^{i\alpha}\phi \approx (1+i\alpha)\phi \Rightarrow \Delta\phi = i\phi; \\ \phi^* \to \phi'^* = e^{-i\alpha}\phi^* \approx (1-i\alpha)\phi^* \Rightarrow \Delta\phi^* = -i\phi^*. \end{aligned} \tag{1.6}$$

We showed in class that the conserved Noether current associated with a symmetry is specified by

$$\begin{aligned} j^\mu &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\Delta\phi^*, \\ &= \left(i\phi\partial^\mu\phi^* - i\phi^*\partial^\mu\phi\right), \\ &= i\left(\phi\partial^\mu\phi^* - \phi^*\partial^\mu\phi\right). \end{aligned} \tag{1.7}$$

2. **a)** The Lagrangian for a source-free electromagnetic field is specified by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \qquad \text{where} \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2.1}$$

It is clear that $F_{\mu\nu}$ is antisymmetric, $F_{\mu\nu} = -F_{\nu\mu}$. From our knowledge of the metric tensor in Minkowski space, it is also clear that $F_{\mu\nu} = -F^{\mu\nu}$ if either μ or ν is zero and $F_{\mu\nu} = F^{\mu\nu}$ if both μ and ν are nonzero. Because the field strength tensor is antisymmetric, our calculation will be much easier.

$$\begin{aligned} \mathcal{L} &= -\frac{1}{2}\left(F_{01}F^{01} + F_{02}F^{02} + F_{03}F^{03} + F_{12}F^{12} + F_{13}F^{13} + F_{23}F^{23}\right), \\ &= \frac{1}{2}\left(F_{01}^2 + F_{02}^2 + F_{03}^2 - F_{12}^2 - F_{13}^2 - F_{23}^2\right), \\ &= \frac{1}{2}[(\partial_0A_1 - \partial_1A_0)^2 + (\partial_0A_2 - \partial_2A_0)^2 + (\partial_0A_3 - \partial_3A_0)^2 \\ &\quad - (\partial_1A_2 - \partial_2A_1)^2 - (\partial_1A_3 - \partial_3A_1)^2 - (\partial_2A_3 - \partial_3A_2)^2], \\ &= \frac{1}{2}\left(\mathbf{E}^2 - \mathbf{B}^2\right). \end{aligned}$$

Now, let us try to find the Euler-Lagrange equations for motion for this field. Note that from our work above if it clear that,

$$\frac{\partial\mathcal{L}}{\partial A_\nu} = 0.$$

After a short while of staring at the above equations, you should see that

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)} &= \begin{cases} (\partial_\mu A_\nu - \partial_\nu A_\mu) & \text{if } \mu = 0 \text{ or } \nu = 0, \\ -(\partial_\mu A_\nu - \partial_\nu A_\mu) & \text{if } \mu,\nu \neq 0, \end{cases} \\ &= -F^{\mu\nu} = F^{\nu\mu}. \end{aligned}$$

So the equations of motion are simply

$$\partial_\mu F^{\nu\mu} = 0. \tag{2.2}$$

Knowing that $E^i = -F^{0i}$ and $\epsilon^{ijk}B^k = F^{ji}$, we can rewrite (2.2) as

$$\begin{aligned} \partial_\mu F^{0\mu} = \partial_i F^{0i} = 0 = -\partial_1E^1 - \partial_2E^2 - \partial_3E^3 = 0, \\ \therefore \nabla\cdot\mathbf{E} = 0. \end{aligned} \tag{2.3}$$

The other equations also can be reduced to familiarity. Specifically,

$$\begin{aligned} \partial_\mu F^{\nu\mu} = \partial_\mu F^{k\mu} = 0, \\ \Longrightarrow \partial_0F^{k0} = \partial_iF^{ki} = \epsilon^{ijk}\partial_iB_j, \\ \therefore \nabla\times\mathbf{B} = \partial_0\mathbf{E}. \end{aligned} \tag{2.4}$$

These two equations represent half of Maxwell's equations for a source-free field. The other two equations relate the vector potential A_ν with the $\mathbf{E}$ and $\mathbf{B}$ fields. These two other equations were 'given.' We needed to know that $\mathbf{B} = \nabla\times\mathbf{A}$ and $\mathbf{E} = -\partial_0\mathbf{A} - \nabla A_0$ to write down the components of $\mathbf{E}$ and $\mathbf{B}$ in terms of $F_{\mu\nu}$.

b) We construct the energy-momentum tensor, $T^{\mu\nu}$, (using the equation derived in my unpublished QFT notes),

$$T^{\mu}_{\ \nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\lambda)}\partial_\nu A_\lambda - \mathcal{L}\delta^{\mu}_{\ \nu}, \tag{2.5}$$

It should be clear that by simply applying our results of part (a)

$$T^{\mu\nu} = F^{\lambda\mu}\partial^\nu A_\lambda - \mathcal{L}\delta^{\mu}_{\ \nu}.$$

This is not symmetric in μ and ν. Remember that the important aspect of $T^{\mu\nu}$ is that it is *conserved*, i.e. $\partial_\mu T^{\mu\nu} = 0$. To make $T^{\mu\nu}$ easier to work with, consider changing it to

$$\hat{T}^{\mu\nu} = T^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu}.$$

Where $K^{\lambda\mu\nu}$ is antisymmetric in its first two indices. By this antisymmetry, it is clear that

$$\partial_\mu \hat{T}^{\mu\nu} = \partial_\mu T^{\mu\nu} + \partial_\mu\partial_\lambda K^{\lambda\mu\nu} = 0.$$

So $\hat{T}^{\mu\nu}$ is a conserved quantity for any $K^{\lambda\mu\nu}$ that is antisymmetric in its first two indices. Let $K^{\lambda\mu\nu} = F^{\mu\lambda}A^\nu$ which is certainly antisymmetric in λ and μ because of $F^{\mu\lambda}$. This allows us to rewrite $\hat{T}^{\mu\nu}$ in a much simpler form. (Note the use of the Euler-Lagrange equations to simplify line 2 below).

$$\begin{aligned}
\hat{T}^{\mu\nu} &= T^{\mu\nu} + \partial_\lambda F^{\mu\lambda}A^\nu, \\
&= T^{\mu\nu} + A^\nu(\partial_\lambda F^{\mu\lambda}) + F^{\mu\lambda}(\partial_\lambda A^\nu), \\
&= T^{\mu\nu} + F^{\mu\lambda}(\partial_\lambda A^\nu), \\
&= F^{\lambda\mu}\partial^\nu A_\lambda + F^{\mu\lambda}\partial_\lambda A^\nu - \mathcal{L}\delta^{\mu}_{\ \nu}, \\
&= F^{\lambda\mu}(\partial^\nu A_\lambda - \partial_\lambda A^\nu) - \mathcal{L}\delta^{\mu}_{\ \nu}.
\end{aligned}$$

It should be clear that $\hat{T}^{\mu\nu} = \hat{T}^{\nu\mu}$. Now we are ready to derive the Hamiltonian and total momentum from $\hat{T}^{\mu\nu}$. First, the Hamiltonian is

$$\begin{aligned}
\mathcal{H} = \mathcal{E} = \hat{T}^{00}&, \\
&= E^i(\partial_i A^0 - \partial^0 A_i) - \mathcal{L}, \\
&= \mathbf{E}^2 - E^i\partial^0 A_i - \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2), \\
&= \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2).
\end{aligned} \tag{2.6}$$

Note that in the last line of the derivation we had to set $E^i\partial^0 A_i = 0$. The total momentum of the field is

$$\begin{aligned}
S^k = T^{0k} &= -E^i(\partial^i A^k - \partial^k A^i), \\
&= E_i(\partial^i A^k - \partial^k A^i), \\
&= E_i\epsilon^{ijk}B_k, \\
\therefore \mathbf{S} &= \mathbf{E}\times\mathbf{B}.
\end{aligned} \tag{2.7}$$

3. **a)** The inner product, (f,g), will be defined

$$(f,g) \equiv i\int d^3x f^*(x)\partial_0 g(x) - g(x)\partial_0 f^*(x),$$

We show that (f,g) is independent of time. This is demonstrated by direct computation.

$$\begin{aligned}\partial_0(f,g) &= i\int d^3x\partial_0\left[f^*(x)\partial_0 g(x) - g(x)\partial_0 f^*(x)\right],\\ &= i\int d^3x\left[\partial_0 f^*(x)\partial_0 g(x) + f^*(x)\partial_0^2 g(x) - g(x)\partial_0^2 f^*(x) - \partial_0 f^*(x)\partial_0 g(x)\right],\\ &= i\int d^3x\left[f^*(x)\partial_0^2 g(x) - g(x)\partial_0^2 f^*(x)\right].\end{aligned}$$

Using the Klein-Gordon equation, this reduces to

$$\begin{aligned}\partial_0(f,g) &= i\int d^3x f^*(\nabla^2 - m^2)g - g(\nabla^2 - m^2)f^*,\\ &= i\int d^3x f^*\nabla^2 g - g\nabla f^*.\end{aligned}$$

We use Green's Theorem to reduce the equation above to

$$\partial_0(f,g) = i\int_S (f^*\nabla g - g\nabla f^*)\vec{n}\cdot da = 0. \tag{3.1}$$

The integral vanishes because we may assume that the fields go to zero at infinity.

b) Recall that the inverse Fourier transform of a Fourier transform of a function is the function itself.

$$f(k) = \int d^3x\left[e^{ikx}\int\frac{d^3k}{(2\pi)^3}e^{-ikx}f(k)\right].$$

Note that when we will express $\phi(x)$ in terms of ladder operators below, ϕ will be a function of the 4-vectors k and x. There is a minus sign to keep track of that is different from the book's 3-vector representation.

$$\phi(x) = \int\frac{d^3k}{(2\pi)^3}\frac{a}{\sqrt{2E_k}}\left(a_k e^{-ikx} + a_k^\dagger e^{ikx}\right).$$

We are now ready to derive the required identity. It will proceed by direct calculation.

$$\begin{aligned}a_k = (f_k(x),\phi(x)) &= i\int d^3x(f^*\partial_0\phi - \phi\partial_0 f^*),\\ &= i\int d^3x\left[e^{ikx}\int\frac{d^3k}{(2\pi)^3}\frac{1}{2E_k}\left(-iE_k a_k e^{-ikx} + iE_k a_k^\dagger e^{ikx}\right)\right.\\ &\qquad\left.-e^{ikx}\int\frac{d^3k}{(2\pi)^3}\frac{iE_k}{2E_k}\left(a_k e^{-ikx} + a_k^\dagger e^{ikx}\right)\right],\\ &= \int d^3x e^{ikx}\left[\int\frac{d^3k}{(2\pi)^3}\frac{1}{2}\left(a_k e^{-ikx} - a_k^\dagger e^{ikx} + a_k e^{-ikx} + a_k^\dagger e^{ikx}\right)\right],\\ &= \int d^3x e^{ikx}\int\frac{d^3k}{(2\pi)^3}e^{-ikx}a_k = a_k,\\ &\therefore a_k = (f_k(x),\phi(x)) = a_k.\end{aligned} \tag{3.2}$$

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c) Let us derive the the commutation relation $\left[a_{\mathbf{p}}, a^\dagger_{\mathbf{p}'}\right] = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}')$. To find this commutation relation, we will first consider the fields in terms of ladder operators.

$$\phi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3}\ \frac{1}{\sqrt{2\omega_{\mathbf{p}}}}(a_{\mathbf{p}} + a^\dagger_{-\mathbf{p}})e^{i\mathbf{p}\cdot\mathbf{x}};$$

$$\pi(\mathbf{y}) = \int \frac{d^3p'}{(2\pi)^3}\ (-i)\sqrt{\frac{\omega_{\mathbf{p}'}}{2}}(a_{\mathbf{p}'} - a^\dagger_{-\mathbf{p}'})e^{i\mathbf{p}'\cdot\mathbf{y}}.$$

Note that because the **p**'s are dummy variables, we cannot assume they are the same when we "mix" the integration, so we have called one **p**'.

$$[\phi(\mathbf{x}),\pi(\mathbf{y})] = i\delta^{(3)}(\mathbf{x}-\mathbf{y})$$

$$= \int \frac{d^3pd^3p'}{(2\pi)^6}\sqrt{\frac{\omega_{\mathbf{p}'}}{\omega_{\mathbf{p}}}}\frac{-i}{2}\left(a_{\mathbf{p}}a_{\mathbf{p}'} - a_{\mathbf{p}}a^\dagger_{-\mathbf{p}'} + a^\dagger_{-\mathbf{p}}a_{\mathbf{p}'} - a^\dagger_{-\mathbf{p}}a^\dagger_{-\mathbf{p}'} - a_{\mathbf{p}'}a_{\mathbf{p}} - a_{\mathbf{p}'}a^\dagger_{-\mathbf{p}} + a^\dagger_{-\mathbf{p}'}a_{\mathbf{p}} + a^\dagger_{-\mathbf{p}'}a^\dagger_{-\mathbf{p}}\right)e^{i(\mathbf{p}\cdot\mathbf{x}+\mathbf{p}'\cdot\mathbf{y})}$$

$$= \int \frac{d^3pd^3p'}{(2\pi)^6}\sqrt{\frac{\omega_{\mathbf{p}'}}{\omega_{\mathbf{p}}}}\frac{i}{2}\left(a_{\mathbf{p}}a^\dagger_{-\mathbf{p}'} - a^\dagger_{-\mathbf{p}'}a_{\mathbf{p}} + a_{\mathbf{p}'}a^\dagger_{-\mathbf{p}} - a^\dagger_{-\mathbf{p}}a_{\mathbf{p}'}\right)e^{i(\mathbf{p}\cdot\mathbf{x}+\mathbf{p}'\cdot\mathbf{y})}\text{(cancelling like terms by symmetry)}$$

$$= \int \frac{d^3pd^3p'}{(2\pi)^6}\sqrt{\frac{\omega_{\mathbf{p}'}}{\omega_{\mathbf{p}}}}\frac{i}{2}\left(\left[a_{\mathbf{p}}, a^\dagger_{-\mathbf{p}'}\right] + \left[a_{\mathbf{p}'}, a^\dagger_{-\mathbf{p}}\right]\right)e^{i(\mathbf{p}\cdot\mathbf{x}+\mathbf{p}'\cdot\mathbf{y})}\quad\left(\text{note that } \left[a_{\mathbf{p}}, a^\dagger_{-\mathbf{p}'}\right] = \left[a_{\mathbf{p}'}, a^\dagger_{-\mathbf{p}}\right]\right)$$

$$= \int \frac{d^3pd^3p'}{(2\pi)^6}\sqrt{\frac{\omega_{\mathbf{p}'}}{\omega_{\mathbf{p}}}}i\left[a_{\mathbf{p}}, a^\dagger_{-\mathbf{p}'}\right]e^{i(\mathbf{p}\cdot\mathbf{x}+\mathbf{p}'\cdot\mathbf{y})} = i\delta^{(3)}(\mathbf{x}-\mathbf{y}). \tag{3.3}$$

Note that by the properties of the Dirac δ functional,

$$\int \frac{d^3pd^3p'}{(2\pi)^3}ie^{i(\mathbf{p}\cdot\mathbf{x}+\mathbf{p}'\cdot\mathbf{y})} = i\delta^{(3)}(\mathbf{x}-\mathbf{y}).$$

Applying this knowledge to (3.3) from above, $\left[a_{\mathbf{p}}, a^\dagger_{-\mathbf{p}'}\right]$ must satisfy

$$\int \frac{d^3pd^3p'}{(2\pi)^3}\sqrt{\frac{\omega_{\mathbf{p}'}}{\omega_{\mathbf{p}}}}[a_{\mathbf{p}}, a^\dagger_{-\mathbf{p}'}] = 1.$$

This is identically satisfied if and only if we have that

$$\left[a_{\mathbf{p}}, a^\dagger_{-\mathbf{p}'}\right] = (2\pi)^3\delta^{(3)}(\mathbf{p}+\mathbf{p}').$$

You can check this statement by noticing that this implies

$$\int \frac{d^3pd^3p'}{(2\pi)^3}\sqrt{\frac{\omega_{\mathbf{p}'}}{\omega_{\mathbf{p}}}}\left[a_{\mathbf{p}}, a^\dagger_{-\mathbf{p}'}\right] = \sqrt{\frac{\omega_{\mathbf{p}}}{\omega_{\mathbf{p}}}} = 1.$$

Therefore, noting our use of $-\mathbf{p}$, we may conclude that

$$\left[a_{\mathbf{p}}, a^\dagger_{\mathbf{p}'}\right] = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}') \tag{3.4}$$

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Physics 513, Quantum Field Theory

Homework 3

Due Tuesday, 23rd September 2003

Jacob Lewis Bourjaily

1. a) We are given complex scalar Lagrangian,

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi.$$

It is clear that the canonical momenta of the field are

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \partial_0 \phi^*;$$
$$\pi^* = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi^*)} = \partial_0 \phi.$$

The canonical commutation relations are then

$$[\phi(x), \partial_0 \phi^*(y)] = [\phi^*(x), \partial_0 \phi(y)] = i\delta^{(3)}(x-y),$$

with all other combinations commuting. As in Homework 2, the Hamiltonian can be directly computed,

$$\begin{aligned} H = \int d^3x \mathcal{H} &= \int d^3x \, (\pi \partial_0 \phi - \mathcal{L}), \\ &= \int d^3x \left(\pi^* \pi - 1/2 \pi^* \pi + 1/2 \nabla \phi^* \nabla \phi + 1/2 m^2 \phi^* \phi\right), \\ &= \frac{1}{2} \int d^3x \left(\pi^* \pi + \nabla \phi^* \nabla \phi + m^2 \phi^* \phi\right). \end{aligned}$$

We can use this expression for the Hamiltonian to find the Heisenberg equation of motion. We have

$$\begin{aligned} i\partial_0 \phi(x) &= \left[\phi(x), \frac{1}{2} \int d^3y \, \left(\pi^*(y)\pi(y) + \nabla \phi^*(y) \nabla \phi(y) + m^2 \phi^*(y)\phi(y)\right)\right], \\ &= \frac{1}{2} \int d^3y \, [\phi(x), \pi(y)] \pi^*(y), \\ &= \frac{i}{2} \int d^3y \, \delta^{(3)}(x-y) \pi^*(y), \\ &= \frac{i}{2} \pi^*(x). \end{aligned}$$

Analogously, $i\partial_0 \phi^*(x) = \frac{i}{2}\pi(x)$. Notice that this derivation used the fact that ϕ commutes with everything in $\mathcal{H}$ except for π. Before we compute the commutator of $\pi^*(x)$ with the Hamiltonian, we should re-write $\mathcal{H}$ as PS did so that our conclusion will be more lucid. We have from above that

$$H = \frac{1}{2} \int d^3x \left(\pi^* \pi + \nabla \phi^* \nabla \phi + m^2 \phi^* \phi\right).$$

We can evaluate the middle term in H using Green's Theorem (essentially integration by parts). We will assume that the surface term vanishes at infinity because the fields must. This allows us to write the Hamiltonian as,

$$H = \frac{1}{2} \int d^3x \left(\pi^* \pi + \phi^*(-\nabla^2 + m^2)\phi\right).$$

Commuting this with $\pi^*(x)$, we conclude that

$$
\begin{aligned}
i\partial_0\pi^*(x) &= \frac{1}{2}\int d^3y\,[\pi^*(x),\phi^*(y)](-\nabla^2+m^2)\phi(y),\\
&= -\frac{i}{2}\int d^3y\,(-\nabla^2+m^2)\phi(y)\delta^{(3)}(x-y),\\
&= -\frac{i}{2}\phi(x).
\end{aligned}
$$

Combining the two results, it is clear that

$$
\begin{aligned}
\partial_0^2\phi(x) &= (\nabla)^2 - m^2)\phi(x),\\
\Longrightarrow (\partial_\mu\partial^\mu + m^2)\phi &= 0.
\end{aligned}
$$

This is just the Klein-Gordon equation. The result is the same for the complex conjugate field.

b) Because the field is no longer purely real, we cannot assume that the coefficient of $e^{i\mathbf{p}\cdot\mathbf{x}}$ in the ladder-operator Fourier expansion is the adjoint of the coefficient of $e^{-i\mathbf{p}\cdot\mathbf{x}}$. Therefore we will use the operator b. The expansion of the fields are then

$$
\begin{aligned}
\phi(x^\mu) &= \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_\mathbf{p}}}\left(a_\mathbf{p}e^{-ip_\mu x^\mu} + b^\dagger_\mathbf{p}e^{ip_\mu x^\mu}\right);\\
\phi^*(x^\mu) &= \int\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2\omega_\mathbf{q}}}\left(a^\dagger_\mathbf{q}e^{iq_\mu x^\mu} + b_\mathbf{q}e^{-iq_\mu x^\mu}\right).
\end{aligned}
$$

It is easy to show that these allow us to define π and π^* in terms of a and b operators as well. These become,

$$
\begin{aligned}
\pi(x^\mu) = \partial_0\phi^*(x^\mu) &= \int\frac{d^3q}{(2\pi)^3}i\sqrt{\frac{\omega_\mathbf{q}}{2}}\left(a^\dagger_\mathbf{q}e^{iq_\mu x^\mu} - b_\mathbf{q}e^{-iq_\mu x^\mu}\right);\\
\pi^*(x^\mu) = \partial_0\phi(x^\mu) &= \int\frac{d^3p}{(2\pi)^3}i\sqrt{\frac{\omega_\mathbf{p}}{2}}\left(-a_\mathbf{p}e^{-ip_\mu x^\mu} + b^\dagger_\mathbf{p}e^{ip_\mu x^\mu}\right).
\end{aligned}
$$

These allow us to directly demonstrate that

$$
\begin{aligned}
[\phi(x^\mu),\pi(y^\mu)] &= \int\frac{d^3pd^3q}{(2\pi)^6}\frac{-i}{2}\sqrt{\frac{\omega_\mathbf{q}}{\omega_\mathbf{p}}}\left(\left[a_\mathbf{p},a^\dagger_\mathbf{q}\right]e^{-i(p_\mu x^\mu - q_\mu x^\mu)} - \left[b^\dagger_\mathbf{p},b_\mathbf{q}\right]e^{i(p_\mu x^\mu - q_\mu x^\mu)}\right),\\
&= i\delta^{(3)}(x-y),
\end{aligned}
$$

while noting that

$$
\left[a_\mathbf{p},a^\dagger_\mathbf{q}\right] = \left[b_\mathbf{p},b^\dagger_\mathbf{q}\right] = (2\pi)^3\delta^{(3)}(p-q),
$$

and all other terms commute. This implies that there are in fact two entirely different sets of particles with the same mass: those created by $b^\dagger$ and those created by $a^\dagger$.

c) I computed the conserved Noether charge in Homework 2 as

$$
j^\mu = i\left(\phi\partial^\mu\phi^* - \phi^*\partial^\mu\phi\right).
$$

We integrate this over all space to see the conserved current in the 0 component. When expressing *phi* and *pi* in terms of ladder operators, we can evaluate this directly.

$$
\begin{aligned}
Q &= \frac{i}{2}\int d^x\left(\phi^*(x)\pi^*(x) - \pi(x)\phi(x)\right),\\
&= \frac{i}{2}\int\frac{d^3xd^3pd^3q}{(2\pi)^6}\left(a_\mathbf{p}a^\dagger_\mathbf{q}e^{ix^\mu(q_\mu-p_\mu)} - a_\mathbf{p}b_\mathbf{q}e^{-ix^\mu(p_\mu+q_\mu)} + b^\dagger_\mathbf{p}a^\dagger_\mathbf{q}e^{ix^\mu(p_\mu+q_\mu)} - b^\dagger_\mathbf{p}b_\mathbf{q}e^{ix^\mu(q_\mu-p_\mu)}\right) - \text{c.c.},\\
&= \frac{i}{2}\int\frac{d^3pd^3q}{(2\pi)^3}\left(a_\mathbf{p}a^\dagger_\mathbf{q}\delta^{(3)}(p-q) - a_\mathbf{p}b_\mathbf{q}\delta^{(3)}(p+q) + b^\dagger_\mathbf{p}a^\dagger_\mathbf{q}\delta^{(3)}(p+q) - b^\dagger_\mathbf{p}b_\mathbf{q}\delta^{(3)}(p-q)\right) - \text{c.c.},\\
&= \frac{i}{2}\int\frac{d^3p}{(2\pi)^3}\left(a_\mathbf{p}a^\dagger_\mathbf{p} - a_\mathbf{p}b_{-\mathbf{p}} + b^\dagger_\mathbf{p}a^\dagger_{-\mathbf{p}} - b^\dagger_\mathbf{p}b_\mathbf{p}\right) - \text{c.c.},\\
&= i\int\frac{d^3p}{(2\pi)^3}\left(a^\dagger_\mathbf{p}a_\mathbf{p} - b^\dagger_\mathbf{p}b_\mathbf{p}\right).
\end{aligned}
$$

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The calculation on the previous page clearly shows that particles that were created by $b^\dagger$ contribute oppositely to those created by $a^\dagger$ to the total charge. We concluded in Homework 2 that this charge was electric charge.

2. a) We are asked to compute the general, K-type Bessel function solution of the Wightman propagator,

$$D_W(x) \equiv \langle 0|\phi(x)\phi(0)|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{-ipx}.$$

Because x is a space-like vector, there exists a reference frame such that $x^0 = 0$. This implies that $x^2 = -\mathbf{x}^2$. And this implies that $px = -\mathbf{p}\cdot\mathbf{x} = -|p||x|\cos(\theta) = -|p|\sqrt{-x^2}\cos(\theta)$. We can then write $D_W(x)$ in polar coordinates as

$$\begin{aligned}
D_W(x) &= \frac{1}{(2\pi)^3}\int_0^{2\pi} d\phi \int_0^{\pi} e^{i|p|\sqrt{-x^2}\cos(\theta)} \int_0^\infty p^2 dp\, \frac{1}{2\sqrt{p^2+m^2}},\\
&= \frac{1}{(2\pi)^2}\int_0^{\pi} d\theta\, e^{i|p|\sqrt{-x^2}\cos(\theta)} \int_0^\infty p^2 dp\, \frac{1}{2\sqrt{p^2+m^2}},\\
&= \frac{1}{(2\pi)^2}\int_{-1}^{1} d\xi\, e^{i|p|\sqrt{-x^2}\xi} \int_0^\infty p^2 dp\, \frac{1}{2\sqrt{p^2+m^2}},\\
&\text{(where } \xi = \cos(\theta))\\
&= \frac{1}{4\pi^2}\int_0^\infty p^2 dp\, \frac{1}{2\sqrt{p^2+m^2}}\frac{1}{i|p|\sqrt{-x^2}}\left(e^{i|p|\sqrt{-x^2}} - e^{-i|p|\sqrt{-x^2}}\right),\\
&= \frac{1}{4\pi^2\sqrt{-x^2}}\int_0^\infty dp\, \frac{p\sin(|p|\sqrt{-x^2})}{\sqrt{p^2+m^2}}.
\end{aligned}$$

Gradsteyn and Ryzhik's equation (3.754.2) states that for a K Bessel function,

$$\int_0^\infty dx\, \frac{\cos(ax)}{\sqrt{\beta^2+x^2}} = K_0(a\beta)).$$

By differentiating both sides with respect to a, it is shown that

$$-\int_0^\infty dx\, \frac{a\sin(ax)}{\sqrt{\beta^2+x^2}} = -\beta K_0'(a\beta) = \beta K_1(a\beta).$$

We can use this identity to write a more concise equation for $D_W(x)$. We may conclude

$$D_W(x) = \frac{m}{4\pi^2\sqrt{-x^2}}K_1(m\sqrt{-x^2}).$$

b) We may compute directly,

$$\begin{aligned}
iD(x) &= \langle 0|[\phi(x),\phi(0)]|0\rangle,\\
&= \langle 0|\phi(x),\phi(0)|0\rangle - \langle 0|\phi(0),\phi(x)|0\rangle,\\
&= D_W(x) - D_W(-x),\\
\Longrightarrow D(x) &= i(D_W(-x) - D_W(x)).
\end{aligned}$$

Similarly,

$$D_1(x) = \langle 0|\{\phi(x),\phi(0)\}|0\rangle = D_W(x) + D_W(-x).$$

It is clear that both function ‘die off’ very rapidly at large distances. I was not able to conclude that they were truly vanishing, but they are certainly nearly-so at even moderately small distances.

Physics 513, Quantum Field Theory

Homework 4

Due Tuesday, 30th September 2003

Jacob Lewis Bourjaily

1. We have defined the *coherent state* by the relation

$$|\{\eta_k\}\rangle \equiv \mathcal{N} \exp\left\{\int \frac{d^3k}{(2\pi)^3} \frac{\eta_k a_k^\dagger}{\sqrt{2E_k}}\right\}|0\rangle.$$

For my own personal convenience throughout this solution, I will let

$$\mathcal{A} \equiv \int \frac{d^3k}{(2\pi)^3} \frac{\eta_k a_k^\dagger}{\sqrt{2E_k}}.$$

a) Lemma: $\left[a_p, e^{\mathcal{A}}\right] = \frac{\eta_p}{\sqrt{2E_p}} e^{\mathcal{A}}$.

proof: First we note that from simple Taylor expansion (which is justified here),

$$e^{\mathcal{A}} = 1 + \mathcal{A} + \frac{\mathcal{A}^2}{2} + \frac{\mathcal{A}^3}{3!} + \ldots$$

Clearly a_p commutes with 1 so we may write,

$$\begin{aligned}
\left[a_p, e^{\mathcal{A}}\right] &= [a_p, \mathcal{A}] + \frac{1}{2}[a_p, \mathcal{A}^2] + \frac{1}{3!}[a_p, \mathcal{A}^3] + \ldots, \\
&= [a_p, \mathcal{A}] + \frac{1}{2}\left([a_p, \mathcal{A}]\mathcal{A} + \mathcal{A}[a_p, \mathcal{A}]\right) + \frac{1}{3!}\left([a_p, \mathcal{A}]\mathcal{A}^2 + \mathcal{A}[a_p, \mathcal{A}]\mathcal{A} + \mathcal{A}[a_p, \mathcal{A}]\mathcal{A}\right) + \ldots, \\
&\overset{*}{=} [a_p, \mathcal{A}]\left(1 + \mathcal{A} + \frac{\mathcal{A}^2}{2} + \frac{\mathcal{A}^3}{3!} + \frac{\mathcal{A}^4}{4!} + \ldots\right), \\
&= [a_p, \mathcal{A}]e^{\mathcal{A}}.
\end{aligned}$$

Note that the step labelled '*' is unjustified. To allow the use of '*' we must show that $[a_p, \mathcal{A}]$ is an invariant scalar and therefore commutes with all the $\mathcal{A}$'s. This is shown by direct calculation.

$$\begin{aligned}
[a_p, \mathcal{A}] &= \int \frac{d^3k}{(2\pi)^3} \frac{\eta_k}{\sqrt{2E_k}} [a_p, a_k^\dagger], \\
&= \int \frac{d^3k}{(2\pi)^3} \frac{\eta_k}{\sqrt{2E_k}} (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{k}), \\
&= \frac{\eta_p}{\sqrt{2E_p}}.
\end{aligned}$$

This proves what was required for '*.' $\frac{\eta_p}{\sqrt{2E_p}}$ is clearly a scalar because η and E_p are real numbers only. But by demonstrating the value of $[a_p, \mathcal{A}]$ we can complete the proof of the required lemma. Clearly,

$$\left[a_p, e^{\mathcal{A}}\right] = [a_p, \mathcal{A}]e^{\mathcal{A}} = \frac{\eta_p}{\sqrt{2E_p}} e^{\mathcal{A}}.$$

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It is clear from the definition of the commutator that $a_p e^{\mathcal{A}} = \left[a_p, e^{\mathcal{A}}\right] + e^{\mathcal{A}} a_p$. Therefore it is intuitively obvious, and also proven that

$$\begin{aligned}
a_p|\{\eta_k\}\rangle &= \mathcal{N} a_p e^{\mathcal{A}}|0\rangle, \\
&= \mathcal{N}\left(\left[a_p, e^{\mathcal{A}}\right] + e^{\mathcal{A}} a_p\right)|0\rangle, \\
&= \mathcal{N}\frac{\eta_p}{\sqrt{2E_p}}|0\rangle + \mathcal{N}e^{\mathcal{A}} a_p|0\rangle, \\
\therefore a_p|\{\eta_k\}\rangle &= \frac{\eta_p}{\sqrt{2E_p}} a_p|\{\eta_k\}\rangle. \qquad (1.1)
\end{aligned}$$

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b) We are to compute the normalization constant $\mathcal{N}$ so that $\langle\{\eta_k\}|\{\eta_k\}\rangle = 1$. I will proceed by direct calculation.

$$\begin{aligned}1 &= \langle\{\eta_k\}|\{\eta_k\}\rangle,\\ &= \mathcal{N}^*\langle 0|e^{\int \frac{d^3k}{(2\pi)^3}\frac{\eta_k a_k}{\sqrt{2E_k}}}|\{\eta_k\}\rangle,\\ &= \mathcal{N}^*\langle 0|e^{\int \frac{d^3k}{(2\pi)^3}\frac{\eta_k}{\sqrt{2E_k}}}|\{\eta_k\}\rangle\end{aligned}$$

because we know that $a_k|\{\eta_k\}\rangle = \frac{\eta_k}{\sqrt{2E_k}}|\{\eta_k\}\rangle$. So clearly

$$\begin{aligned}1 &= |\mathcal{N}|^2 e^{\int \frac{d^3k}{(2\pi)^3}\frac{\eta_k^2}{2E_k}},\\ \therefore \mathcal{N} &= e^{-\frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\frac{\eta_k^2}{2E_k}}.\end{aligned}$$

c) We will find the expectation value of the field $\phi(x)$ by direct calculation as before.

$$\begin{aligned}\overline{\phi(x)} = \langle\{\eta_k\}|\phi(x)|\{\eta_k\}\rangle &= \langle\{\eta_k\}|\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(a_p e^{i\vec{p}\cdot\vec{x}} + a_p^\dagger e^{-i\vec{p}\cdot\vec{x}}\right)|\{\eta_k\}\rangle,\\ &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(\underbrace{\langle\{\eta_k\}|a_p e^{i\vec{p}\cdot\vec{x}}|\{\eta_k\}\rangle}_{\text{act with } a_p \text{ to the right}} + \underbrace{\langle\{\eta_k\}|a_p^\dagger e^{-i\vec{p}\cdot\vec{x}}|\{\eta_k\}\rangle}_{\text{act with } a_p^\dagger \text{ to the left}}\right),\\ &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(\frac{\eta_p}{\sqrt{2E_p}}e^{i\vec{p}\cdot\vec{x}} + \frac{\eta_p}{\sqrt{2E_p}}e^{-i\vec{p}\cdot\vec{x}}\right),\\ &= \int \frac{d^3p}{(2\pi)^3}\frac{\eta_p}{E_p}\cos(\vec{p}\cdot\vec{x}).\end{aligned}$$

d) We will compute the expected particle number directly.

$$\begin{aligned}\overline{N} = \langle\{\eta_k\}|N|\{\eta_k\}\rangle &= \langle\{\eta_k\}|\int \frac{d^3p}{(2\pi)^3}a_p^\dagger a_p|\{\eta_k\}\rangle,\\ &= \int \frac{d^3p}{(2\pi)^3}\left(\langle\{\eta_k\}|\underleftarrow{a_p^\dagger}\ \underrightarrow{a_p}|\{\eta_k\}\rangle\right),\\ &= \int \frac{d^3p}{(2\pi)^3}\frac{\eta_p^2}{2E_p}.\end{aligned}$$

e) To compute the mean square dispersion, let us recall the theorem of elementary probability theory that

$$\langle(\Delta N)^2\rangle = \overline{N^2} - \overline{N}^2.$$

We have already calculated $\overline{N}$ so it is trivial to note that

$$\overline{N}^2 = \int \frac{d^3k d^3p}{(2\pi)^6}\frac{\eta_k^2\eta_p^2}{4E_kE_p}.$$

Let us then calculate $\overline{N^2}$.

$$\begin{aligned}\overline{N^2} = \langle\{\eta_k\}|N^2|\{\eta_k\}\rangle &= \langle\{\eta_k\}|\int \frac{d^3k d^3p}{(2\pi)^6}a_k^\dagger a_k a_p^\dagger a_p|\{\eta_k\}\rangle,\\ &= \int \frac{d^3k d^3p}{(2\pi)^6}\frac{\eta_k\eta_p}{2\sqrt{E_kE_p}}\langle\{\eta_k\}|a_k a_p^\dagger|\{\eta_k\}\rangle,\\ &= \int \frac{d^3k d^3p}{(2\pi)^6}\frac{\eta_k\eta_p}{2\sqrt{E_kE_p}}\left((2\pi)^3\delta^{(3)}(\vec{k}-\vec{p}) + \langle\{\eta_k\}|a_p^\dagger a_k|\{\eta_k\}\rangle\right),\\ &= \int \frac{d^3k}{(2\pi)^3}\frac{\eta_k^2}{2E_k} + \int \frac{d^3k d^3p}{(2\pi)^6}\frac{\eta_k^2\eta_p^2}{4E_kE_p}.\end{aligned}$$

It is therefore quite easy to see that

$$\langle(\Delta N)^2\rangle = \overline{N^2} - \overline{N}^2 = \int \frac{d^3k}{(2\pi)^3}\frac{\eta_k^2}{2E_k}.$$

2. We are given the Lorentz commutation relations,

$$[J^{\mu\nu}, J^{\rho\sigma}] = i(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}).$$

a) Given the generators of rotations and boosts defined by,

$$L^i = \frac{1}{2}\epsilon^{ijk}J^{jk} \qquad K^i = J^{0i},$$

we are to explicitly calculate all the commutation relations. We are given trivially that

$$[L^i, L^j] = i\epsilon^{ijk}L^k.$$

Let us begin with the K's. By direct calculation,

$$\begin{aligned}[K^i, K^j] = [J^{0i}, J^{0j}] &= i(g^{0i}J^{0j} - g^{00}J^{ij} - g^{ij}J^{00} + g^{0j}J^{i0}),\\ &= -iJ^{ij};\\ &= -2i\epsilon^{ijk}L^k.\end{aligned}$$

Likewise, we can directly compute the commutator between the L and K's. This also will follow by direct calculation.

$$\begin{aligned}[L^i, K^j] &= \frac{1}{2}\epsilon^{lk}[J^{ilk}, J^{0j}],\\ &= \frac{1}{2}\epsilon^{ilk}i(g^{l0}J^{ij} - g^{i0}J^{lj} - g^{lj}J^{i0} + g^{ij}J^{l0}),\\ &= i\epsilon^{ijk}J^{0k};\\ &= i\epsilon^{ijk}K^k.\end{aligned}$$

We were also to show that the operators

$$J_+^i = \frac{1}{2}(L^i + iK^i) \qquad J_-^i = \frac{1}{2}(L^i - iK^i),$$

could be seen to satisfy the commutation relations of angular momentum. First let us compute,

$$\begin{aligned}[J_+, J_-] &= \frac{1}{4}\left[(L^i + iK^i), (L^j - iK^i)\right],\\ &= \frac{1}{4}\left([L^i, L^j] + i[K^i, L^j] - i[L^i, K^j] + [K^i, K^j]\right),\\ &= 0.\end{aligned}$$

In the last line it was clear that I used the commutator $[L^i, K^j]$ derived above. The next two calculations are very similar and there is a lot of 'justification' algebra in each step. There is essentially no way for me to include all of the details of every step, but each can be verified (e.g. $i[K^i, L^j] = -i[L^j, K^i] = (-i)i\epsilon^{jik}K^k = -\epsilon^{ijk}K^k$*...etc*). They are as follows:

$$\begin{aligned}[J_+^i, J_+^j] &= \frac{1}{4}\left[(L^i + iK^i), (L^j + iK^j)\right],\\ &= \frac{1}{4}\left([L^i, L^j] + i[K^i, L^j] + i[L^i, K^j] + i[L^i, K^i] - [K^i, K^j]\right),\\ &= \frac{1}{4}\left(i\epsilon^{ijk}L^k - \epsilon^{ijk}K^k - \epsilon^{ijk}K^k + i\epsilon^{ijk}L^k\right),\\ &= i\epsilon^{ijk}\frac{1}{2}(L^k + iK^k) = i\epsilon^{ijk}J_+^k.\end{aligned}$$

Likewise,

$$\begin{aligned}[J_-^i, J_-^j] &= \frac{1}{4}\left[(L^i - iK^i), (L^j - iK^j)\right],\\ &= \frac{1}{4}\left([L^i, L^j] - i[K^i, L^j] - i[L^i, K^j] + i[L^i, K^i] - [K^i, K^j]\right),\\ &= \frac{1}{4}\left(i\epsilon^{ijk}L^k + \epsilon^{ijk}K^k + \epsilon^{ijk}K^k + i\epsilon^{ijk}L^k\right),\\ &= i\epsilon^{ijk}\frac{1}{2}(L^k - iK^k) = i\epsilon^{ijk}J_-^k.\end{aligned}$$

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b) Let us consider first the $(0, \frac{1}{2})$ representation. For this representation we will need to satisfy

$$J_+^i = \frac{1}{2}(L^i + iK^i) = 0 \qquad J_-^i = \frac{1}{2}(L^i - iK^k) = \frac{\sigma^i}{2}.$$

This is obtained by taking $L^i = \frac{\sigma^i}{2}$ and $K^i = \frac{i\sigma^i}{2}$. The transformation law then of the $(0, \frac{1}{2})$ representation is

$$\begin{aligned}\Phi_{(0,\frac{1}{2})} &\longrightarrow e^{-i\omega_{\mu\nu}J^{\mu\nu}}\Phi_{(0,\frac{1}{2})},\\ &= e^{-i(\theta^i L^i + \beta^j K^j)}\Phi_{(0,\frac{1}{2})},\\ &= e^{-\frac{i\theta^i\sigma^i}{2} + \frac{\beta^j K^j}{2}}\Phi_{(0,\frac{1}{2})}.\end{aligned}$$

The calculation for the $(\frac{1}{2}, 0)$ representation is very similar. Taking $L^i = \frac{\sigma^i}{2}$ and $K^i = -\frac{\sigma^i}{2}$, we get

$$J_+^i = \frac{1}{2}(L^i + iK^i) = \frac{\sigma^i}{2} \qquad J_-^i = \frac{1}{2}(L^i - iK^k) = 0.$$

Then the transformation law of the representation is

$$\begin{aligned}\Phi_{(\frac{1}{2},0)} &\longrightarrow e^{-i\omega_{\mu\nu}J^{\mu\nu}}\Phi_{(\frac{1}{2},0)},\\ &= e^{-i(\theta^i L^i + \beta^j K^j)}\Phi_{(\frac{1}{2},0)},\\ &= e^{-\frac{i\theta^i\sigma^i}{2} - \frac{\beta^j K^j}{2}}\Phi_{(\frac{1}{2},0)}.\end{aligned}$$

Comparing these transformation laws with Peskin and Schroeder's (3.37), we see that

$$\psi_L = \Phi_{(\frac{1}{2},0)} \qquad \psi_R = \Phi_{(0,\frac{1}{2})}.$$

3. a) We are given that T_a is a representation of some Lie group. This means that

$$[T_a, T_b] = if^{abc}T_c$$

by definition. Allow me to take the complex conjugate of both sides. Note that $[T_a, T_b] = [(-T_a), (-T_b)]$ in general and recall that f^{abc} are real.

$$\begin{aligned}[T_a, T_b]^* &= (if^{abc}T_c)^*,\\ [T_a^*, T_b^*] &= -if^{abc}T_c^*,\\ \therefore [(-T_a^*), (-T_b^*)] &= if^{abc}(-T_c^*).\end{aligned}$$

So by the definition of a representation, it is clear that $(-T_a^*)$ is also a representation of the algebra.

b) As before, we are given that T_a is a representation of some Lie group. We will take the Hermitian adjoint of both sides.

$$\begin{aligned}[T_a, T_b]^\dagger &= (if^{abc}T_c)^\dagger,\\ (T_aT_b)^\dagger - (T_bT_a)^\dagger &= -if^{abc}T_c^\dagger,\\ T_b^\dagger T_a^\dagger - T_a^\dagger T_b^\dagger &= -if^{abc}T_c^\dagger,\\ [T_b^\dagger, T_a^\dagger] &= -if^{abc}T_c^\dagger,\\ \therefore [T_a^\dagger, T_b^\dagger] &= if^{abc}T_c^\dagger.\end{aligned}$$

So by the definition of a representation, it is clear that $T_a^\dagger$ is a representation of the algebra.

c) We define the spinor representation of $SU(2)$ by $T_a = \frac{\sigma^a}{2}$ so that

$$T_1 \equiv \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad T_2 \equiv \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad T_3 \equiv \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We will consider the matrix $S = i\sigma^2$. Clearly S is unitary because $(i\sigma^2)(i\sigma^2)^\dagger = 1$. Now, one could proceed by direct calculation to demonstrate that

$$ST_1S^\dagger = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = -T_1^* \qquad ST_2S^\dagger = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = -T_2^* \qquad ST_3S^\dagger = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -T_3^*.$$

This clearly demonstrates that the representation $-T_a^*$ is equivalent to that of T_a.

d) From our definitions of our representation of $SO(3,1)$ using J_+^i and J_-^i, it is clear that

$$(J_+^i)^\dagger = J_-^i.$$

This could be expressed as if $(\frac{1}{2},0)^\dagger = (0,\frac{1}{2})$, or, rather $L^\dagger = R$. So what we must ask ourselves is, does there exist a unitary matrix S such that

$$SLS^\dagger = L \qquad \text{but} \qquad SKS^\dagger = -K\ ?$$

If there did exist such a unitary transformation, then we could conclude that L and R are equivalent representations. However, this is not possible in our $SO(3,1)$ representation because both L and K are represented strictly by the Pauli spin matrices so that $iK = L = \frac{\sigma}{2}$. It is therefore clear that there cannot exist a transformation that will change the sign of K yet leave L alone. So the representations are inequivalent.

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Physics 513, Quantum Field Theory

Homework 5

Due Tuesday, 7th October 2003

Jacob Lewis Bourjaily

1. We are to verify the identity

$$[\gamma^\mu, S^{\rho\sigma}] = (\mathcal{J}^{\rho\sigma})^\mu{}_\nu \gamma^\nu.$$

It will be helpful to first have a good representation of $(\mathcal{J}^{\rho\sigma})^\mu{}_\nu$. This can be obtained by raising one of the indices of $(\mathcal{J}^{\rho\sigma})_{\lambda\nu}$ which is defined in Peskin and Schroeder's equation 3.18.

$$\begin{aligned}(\mathcal{J}^{\rho\sigma})^\mu{}_\nu = g^{\mu\lambda}(\mathcal{J}^{\rho\sigma})_{\lambda\nu} &= ig^{\mu\lambda}(\delta^\rho_\lambda\delta^\sigma_\nu - \delta^\rho_\nu\delta^\sigma_\lambda),\\ &= i(g^{\mu\rho}\delta^\sigma_\nu - g^{\mu\sigma}\delta^\rho_\nu).\end{aligned}$$

We will use this expression for $(\mathcal{J}^{\rho\sigma})^\mu{}_\nu$ in the last line of our derivation below. We will proceed by direct computation.

$$\begin{aligned}[\gamma^\mu, S^{\rho\sigma}] &= \frac{i}{4}\left([\gamma^\mu,\gamma^\rho\gamma^\sigma] - [\gamma^\mu,\gamma^\sigma\gamma^\rho]\right),\\ &= \frac{i}{4}\left(\{\gamma^\mu,\gamma^\rho\}\gamma^\sigma - \gamma^\rho\{\gamma^\mu,\gamma^\sigma\} - \{\gamma^\mu,\gamma^\sigma\}\gamma^\rho + \gamma^\sigma\{\gamma^\mu,\gamma^\rho\}\right),\\ &= \frac{i}{2}\left(g^{\mu\rho}\gamma^\sigma - \gamma^\rho g^{\mu\sigma} - g^{\mu\sigma}\gamma^\rho + \gamma^\sigma g^{\mu\rho}\right),\\ &= i\left(g^{\mu\rho}\gamma^\sigma - g^{\mu\sigma}\gamma^\rho\right),\\ &= i\left(g^{\mu\rho}\delta^\sigma_\nu\gamma^\nu - g^{\mu\sigma}\delta^\rho_\nu\gamma^\nu\right),\\ &= i\left(g^{\mu\rho}\delta^\sigma_\nu - g^{\mu\sigma}\delta^\rho_\nu\right)\gamma^\nu,\\ \therefore [\gamma^\mu, S^{\rho\sigma}] &= (\mathcal{J}^{\rho\sigma})^\mu{}_\nu\gamma^\nu.\end{aligned}$$

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2. All of the required identities will be computed by directly.

a) $\gamma_\mu\gamma^\mu = 4$

$$\gamma_\mu\gamma^\mu = (\gamma^0)^2 + (\gamma^1)^2 + (\gamma^2)^2 + (\gamma^3)^2 = 4.$$

b) $\gamma_\mu \not{k}\gamma^\mu = -2\not{k}$

$$\begin{aligned}\gamma_\mu\not{k}\gamma^\mu &= \gamma_\mu\gamma_\nu k^\nu\gamma^\mu,\\ &= (2g_{\mu\nu} - \gamma_\nu\gamma_\mu)k^\nu\gamma^\mu,\\ &= 2k_\mu\gamma^\mu - \gamma_\nu k^\nu\gamma_\mu\gamma^\mu,\\ \therefore \gamma_\mu\not{k}\gamma^\mu &= -2\not{k}\end{aligned}$$

c) $\gamma_\mu\not{p}\not{q}\gamma^\mu = 4p\cdot q$

$$\begin{aligned}\gamma_\mu\not{p}\not{q}\gamma^\mu &= \gamma_\mu\gamma_\nu p^\nu q_\rho\gamma^\rho\gamma^\mu,\\ &= (2g_{\mu\nu} - \gamma_\nu\gamma_\mu)p^\nu q_\rho(2g^{\rho\mu} - \gamma^{\mu\rho}),\\ &= (2p_\mu - \not{p}\gamma_\mu)(2q^\mu - \not{q}\gamma^\mu),\\ &= 4p\cdot q - 2\not{p}\not{q} - 2\not{p}\not{q} + 4\not{p}\not{q},\\ \therefore \gamma_\mu\not{p}\not{q}\gamma^\mu &= 4p\cdot q.\end{aligned}$$

d) $\gamma_\mu\not{k}\not{p}\not{q}\gamma^\mu = -2\not{p}\not{q}\not{k}$

By repeated use of the identity $\gamma^\mu\gamma^\nu = 2g^{\mu\nu} - \gamma^\nu\gamma^\mu$,

$$\begin{aligned}\gamma_\mu\not{k}\not{p}\not{q}\gamma^\mu &= \gamma_\mu\gamma^\nu k_\nu\gamma^\rho p_\rho\gamma^\sigma q_\sigma\gamma^\mu,\\ &= 2\gamma_\mu\not{k}\not{p}q_\sigma g^{\sigma\mu} - 2\gamma_\mu\not{k}p_\rho g^{\rho\mu}\not{q} + 2\gamma_\mu k_\nu g^{\nu\mu}\not{p}\not{q} - 4\not{k}\not{p}\not{q},\\ &= 2\not{q}\not{k}\not{p} - 2\not{p}\not{k}\not{q} - 2\not{k}\not{p}\not{q},\\ &= 4\not{q}k\cdot p - 2\not{q}\not{p}\not{k} - 4p\cdot k\not{q},\\ \therefore \gamma_\mu\not{k}\not{p}\not{q}\gamma^\mu &= -2\not{p}\not{q}\not{k}.\end{aligned}$$

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3. We are to prove the Gordon identity,

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left[\frac{(p'+p)^\mu}{2m} + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\right]u(p).$$

Explicitly writing out each term in the brackets and recalling the anticommutation relations of γ^μ, the right hand side becomes,

$$\begin{aligned}\bar{u}(p')\left[\frac{(p'+p)^\mu}{2m} + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\right]u(p) &= \bar{u}(p')\left[\frac{1}{2m}\left(p'^\mu + p^\mu - \tfrac{1}{2}\gamma^\mu\gamma^\nu(p-p')_\nu + \tfrac{1}{2}\gamma^\nu\gamma^\mu(p-p')_\nu\right)\right]u(p),\\ &= \bar{u}(p')\left[\frac{1}{2m}\left(p'^\mu + p^\mu - \tfrac{1}{2}\gamma^\mu\gamma^\nu(p-p')_\nu + g^{\nu\mu}(p-p')_\nu - \tfrac{1}{2}\gamma^\mu\gamma^\nu(p-p')_\nu\right)\right]u(p),\\ &= \bar{u}(p')\left[\frac{1}{2m}\left(2p'^\mu - \gamma^\mu\gamma^\nu(p-p')_\nu\right)\right]u(p),\\ &= \bar{u}(p')\left[\frac{1}{2m}\left(2p'^\mu - \gamma^\mu\not{p} - \gamma^\mu\not{p}'\right)\right]u(p).\end{aligned}$$

Now, recall that the Dirac equation for $u(p)$ is

$$\not{p}u(p) = mu(p).$$

Converting this for $\bar{u}(p')\not{p}'$, one obtains

$$\bar{u}(p')\not{p}' = m\bar{u}(p').$$

Applying both of these equations where we left of, we see that

$$\bar{u}(p')\left[\frac{(p'+p)^\mu}{2m} + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\right]u(p) = \bar{u}(p')\frac{p'^\mu}{m}u(p).$$

Looking again at the Dirac equation, $m\bar{u}(p') = \bar{u}(p')\not{p}' = \bar{u}(p')\gamma^\mu p'_\mu$, it is clear that

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left[\frac{(p'+p)^\mu}{2m} + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\right]u(p).$$

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4. **a)** To demonstrate that $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$ anticommutes each of the γ^μ, it will be helpful to remember that whenever $\mu \neq \nu$, $\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu$ by the anticommutation relations. Therefore, any odd permutation in the order of some γ's will change the sign of the expression. It should therefore be quite clear that

$$\begin{aligned}\gamma^5\gamma^0 &= i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0 = -i\gamma^1\gamma^2\gamma^3 = -i\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma^0\gamma^5;\\ \gamma^5\gamma^1 &= i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^1 = i\gamma^0\gamma^2\gamma^3 = -i\gamma^1\gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma^1\gamma^5;\\ \gamma^5\gamma^2 &= i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^2 = -i\gamma^0\gamma^1\gamma^3 = -i\gamma^2\gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma^2\gamma^5;\\ \gamma^5\gamma^3 &= i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^3 = i\gamma^0\gamma^1\gamma^2 = -i\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma^3\gamma^5;\\ &\therefore \{\gamma^5, \gamma^\mu\} = 0.\end{aligned}$$

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b) We will first show that γ^5 is hermitian. Note that the derivation relies on the fact that $(\gamma^0)^\dagger = \gamma^0$ and $(\gamma^i)^\dagger = -\gamma^i$. These facts are inherent in our chosen representation of the γ matrices.

$$\begin{aligned}(\gamma^5)^\dagger &= -i(\gamma^0\gamma^1\gamma^2\gamma^3)^\dagger,\\ &= -i(\gamma^3)^\dagger(\gamma^2)^\dagger(\gamma^1)^\dagger(\gamma^0)^\dagger,\\ &= i\gamma^3\gamma^2\gamma^1\gamma^0,\\ &= -i\gamma^2\gamma^1\gamma^0\gamma^3,\\ &= -i\gamma^1\gamma^0\gamma^2\gamma^3,\\ &= i\gamma^0\gamma^1\gamma^2\gamma^3,\\ &= \gamma^5.\end{aligned}$$

Let us now show that $(\gamma^5)^2 = 1$.

$$\begin{aligned}(\gamma^5)^2 &= -i\gamma_3\gamma_2\gamma_1\gamma_0 i\gamma^0\gamma^1\gamma^2\gamma^3,\\ &= \gamma_3\gamma_2\gamma_1\gamma_0\gamma^0\gamma^1\gamma^2\gamma^3,\\ &= \gamma_3\gamma_2\gamma_1\gamma^1\gamma^2\gamma^3,\\ &= \gamma_3\gamma_2\gamma^2\gamma^3,\\ &= \gamma_3\gamma^3,\\ &= 1.\end{aligned}$$

c) Note that $\epsilon_{\kappa\lambda\mu\nu}$ is only nonzero when $\kappa \neq \lambda \neq \mu \neq \nu$ which leaves exactly $4! = 24$ nonzero terms from the 24 possible permutations. Also note that $\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu$, like $\epsilon_{\kappa\lambda\mu\nu}$, is totally antisymmetric–any odd permutation of indices changes the sign of the argument. Therefore, they change sign exactly together, $\epsilon_{\kappa\lambda\mu\nu}\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu$ does not change sign. That is to say that each of the 24 nonzero terms of $\epsilon_{\kappa\lambda\mu\nu}\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu$ is identical to $\epsilon_{0123}\gamma^0\gamma^1\gamma^2\gamma^3$. So

$$\epsilon_{\kappa\lambda\mu\nu}\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu = 24\epsilon_{0123}\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{24}{i}\gamma^5,$$

$$\therefore \gamma^5 = -\frac{i}{24}\epsilon_{\kappa\lambda\mu\nu}\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu.$$

This implies that

$$\gamma^5 = -i\epsilon_{\kappa\lambda\mu\nu}\gamma^{[\kappa}\gamma^\lambda\gamma^\mu\gamma^{\nu]},$$

$$\therefore \gamma^{[\kappa}\gamma^\lambda\gamma^\mu\gamma^{\nu]} = -i\epsilon^{\kappa\lambda\mu\nu}\gamma^5.$$

5. We will begin by simply directly computing the form of $\xi_\pm$ from the eigenvalue equation

$$(\hat{\mathbf{p}}\cdot\tfrac{1}{2}\vec{\sigma})\,\xi_\pm(\hat{\mathbf{p}}) = \pm\tfrac{1}{2}\xi_\pm(\hat{\mathbf{p}}).$$

Let us begin to expand the left hand side of the eigenvalue equation,

$$(\hat{\mathbf{p}}\cdot\tfrac{1}{2}\vec{\sigma}) = \frac{1}{2}\begin{pmatrix} 0 & \sin\theta\cos\phi \\ \sin\theta\cos\phi & 0\end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & -i\sin\theta\sin\phi \\ i\sin\theta\sin\phi & 0\end{pmatrix} + \frac{1}{2}\begin{pmatrix} \cos\theta & 0 \\ 0 & -\cos\theta\end{pmatrix},$$

$$\therefore\ (\hat{\mathbf{p}}\cdot\tfrac{1}{2}\vec{\sigma}) = \frac{1}{2}\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta\end{pmatrix}.$$

Note that we can see here that because this matrix has determinant -1 and trace 0, the eigenvalues must be are ± 1. Therefore, we may write the eigenvalue equation as the system of equations,

$$\frac{1}{2}\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta\end{pmatrix}\begin{pmatrix}\xi^1_\pm \\ \xi^2_\pm\end{pmatrix} = \pm\frac{1}{2}\begin{pmatrix}\xi^1_\pm \\ \xi^2_\pm\end{pmatrix}.$$

These two equations are equivalent; I will use the first row of equations. This becomes

$$\pm\xi^1_\pm = \cos\theta\xi^1_\pm + \sin\theta e^{-i\phi}\xi^2_\pm.$$

Therefore,

$$\xi^1_+ = \frac{\sin\theta e^{-i\phi}\xi^2_+}{1-\cos\theta} = e^{-i\phi}\tan(\theta/2)\xi^2_+ \qquad \text{and} \qquad \xi^1_- = -\frac{\sin\theta e^{-i\phi}\xi^2_-}{1+\cos\theta} = -e^{-i\phi}\tan(\theta/2)\xi^2_-$$

So that

$$\xi_+ = \begin{pmatrix} e^{-i\phi}\cot(\theta/2)\xi^2_+ \\ \xi^2_+\end{pmatrix} \qquad \text{and} \qquad \xi_- = \begin{pmatrix} -e^{-i\phi}\tan(\theta/2)\xi^2_- \\ \xi^2_-\end{pmatrix}.$$

To find the normalization, we must apply the normalization conditions $\xi^\dagger_\pm\xi_\pm = 1$. By direct calculation,

$$\begin{aligned}\xi^\dagger_+\xi_+ = 1 &= (\xi^2_+)^2(\cot^2(\theta/2)+1),\\ &= \frac{(\xi^2_+)^2}{\sin^2(\theta/2)},\\ \therefore \xi^2_+ &= e^{i\eta^+}\sin(\theta/2).\end{aligned}$$

Likewise for ξ_-,

$$\begin{aligned}\xi_-^\dagger \xi_- = 1 &= (\xi_-^2)^2(\tan^2(\theta/2)+1),\\ &= \frac{(\xi_-^2)^2}{\cos^2(\theta/2)},\\ \therefore \xi_-^2 &= e^{i\eta^-}\cos(\theta/2).\end{aligned}$$

Notice that if ξ_+ satisfies $\xi^\dagger\xi = 1$ then so does $\xi' = e^{i\eta}\xi$. So in solving the normalization equations, we necessarily introduced an arbitrary phase η. Noting, this, spinors become

$$\xi_+ = e^{i\eta^+}\begin{pmatrix} e^{-i\phi}\cos(\theta/2)\\ \sin(\theta/2)\end{pmatrix} \qquad \text{and} \qquad \xi_- = e^{i\eta^-}\begin{pmatrix} -e^{-i\phi}\sin(\theta/2)\\ \cos(\theta/2)\end{pmatrix}.$$

Lastly, we would like to set the phase η so that when the particle is moving in the $+z-$direction, they reduce to the usual spin-up/spin-down forms. It should be quite obvious that $\eta^- = 0$ satisfies this condition for ξ_-. For ξ^+, we will set the phase to $\eta^+ = \phi$ so that we may lose the $e^{-i\phi}$ term when $\theta = 0$. So we may write our final spinors as

$$\xi_+ = \begin{pmatrix} \cos(\theta/2)\\ e^{i\phi}\sin(\theta/2)\end{pmatrix} \qquad \text{and} \qquad \xi_- = \begin{pmatrix} -e^{-i\phi}\sin(\theta/2)\\ \cos(\theta/2)\end{pmatrix}.$$

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Physics 513, Quantum Field Theory

Homework 6

Due Tuesday, 21st October 2003

Jacob Lewis Bourjaily

1. For the following derivations it will be helpful to recall the following:

$$\begin{aligned}\mathcal{P}\psi(t,\vec{x})\mathcal{P}^\dagger &= \eta_a\gamma^0\psi(t,-\vec{x});\\ \mathcal{P}\bar{\psi}(t,\vec{x})\mathcal{P}^\dagger &= \eta_a^*\bar{\psi}(t,-\vec{x})\gamma^0;\\ \mathcal{C}\psi\mathcal{C}^\dagger &= -i(\bar{\psi}\gamma^0\gamma^2)^\intercal;\\ \mathcal{C}\bar{\psi}\mathcal{C}^\dagger &= -i(\gamma^0\gamma^2\psi)^\intercal;\\ \mathcal{T}\psi(t,\vec{x})\mathcal{T}^\dagger &= \gamma^1\gamma^3\psi(-t,\vec{x});\\ \mathcal{T}\bar{\psi}(t,\vec{x})\mathcal{T}^\dagger &= -\bar{\psi}(-t,\vec{x})\gamma^1\gamma^3.\end{aligned}$$

a) We are to verify the transformation properties of $A^\mu \equiv \bar{\psi}\gamma^\mu\gamma^5\psi$ and $T^{\mu\nu} \equiv \bar{\psi}\sigma^{\mu\nu}\psi$ under $\mathcal{P}$.

Let us first consider the axial vector A^μ.

$$\begin{aligned}\mathcal{P}A^\mu\mathcal{P}^\dagger = \mathcal{P}\bar{\psi}\gamma^\mu\gamma^5\psi\mathcal{P}^\dagger &= \eta_a^*\bar{\psi}\gamma^0\gamma^\mu\gamma^5\eta_a\gamma^0\psi,\\ &= \bar{\psi}\gamma^0\gamma^\mu\gamma^5\gamma^0\psi,\\ &= -\bar{\psi}\gamma^0\gamma^\mu\gamma^0\gamma^5\psi,\\ &= -\bar{\psi}\gamma_\mu\gamma^5\psi = -A_\mu.\end{aligned}$$

The last step can be seen by noticing that

$$\gamma^0\gamma^\mu\gamma^0 = \left\{\begin{array}{cc}\gamma^\mu & \mu = 0\\ -\gamma^\mu & \mu = 1,2,3\end{array}\right\} = \gamma_\mu.$$

Now we will consider the transformation of the tensor $T^{\mu\nu}$.

$$\begin{aligned}\mathcal{P}T^{\mu\nu}\mathcal{P}^\dagger = \mathcal{P}\bar{\psi}\sigma^{\mu\nu}\psi\mathcal{P}^\dagger &= \eta_a^*\bar{\psi}\gamma^0\sigma^{\mu\nu}\eta_a\gamma^0\psi,\\ &= \bar{\psi}\gamma^0\sigma^{\mu\nu}\gamma^0\psi,\\ &= \bar{\psi}\sigma_{\mu\nu}\psi = T_{\mu\nu}.\end{aligned}$$

Similar to the axial vector case, the last step is a result of directly verifying the identity

$$\gamma^0\sigma^{\mu\nu}\gamma^0 = \frac{i}{2}(\gamma^0\gamma^\mu\gamma^\nu\gamma^0 - \gamma^0\gamma^\nu\gamma^\mu\gamma^0) = \left\{\begin{array}{cc}\sigma^{\mu\nu} & \mu,\nu\neq 0 \text{ or } \mu,\nu = 0\\ -\sigma^{\mu\nu} & \text{one of } \mu \text{ or } \nu = 0\end{array}\right\} = \sigma_{\mu\nu}.$$

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b) We are to verify the transformation properties of $V^\mu \equiv \bar{\psi}\gamma^\mu\psi$ and $A^\mu \equiv \bar{\psi}\gamma^\mu\gamma^5\psi$ under $\mathcal{C}$.

Let us first consider the transformation of the vector V^μ.

$$\begin{aligned}\mathcal{C}V^\mu\mathcal{C}^\dagger = \mathcal{C}\bar{\psi}\gamma^\mu\psi\mathcal{C}^\dagger &= -i(\gamma^0\gamma^2)^\intercal\gamma^\mu(-i)(\bar{\psi}\gamma^0\gamma^2)^\intercal,\\ &= -\bar{\psi}\gamma^0\gamma^2\gamma^{\mu\intercal}\gamma^0\gamma^2\psi,\\ &= \bar{\psi}\gamma^0\gamma^2\gamma^{\mu\intercal}\gamma^2\gamma^0\psi,\\ &= -\bar{\psi}\gamma^\mu\psi = -V^\mu.\end{aligned}$$

Let us now consider the axial vector A^μ.

$$\begin{aligned}\mathcal{C}A^\mu\mathcal{C}^\dagger = \mathcal{C}\bar{\psi}\gamma^\mu\gamma^5\psi\mathcal{C}^\dagger &= -i(\gamma^0\gamma^2\psi)^\intercal\gamma^\mu\gamma^5(-i)(\bar{\psi}\gamma^0\gamma^2)^\intercal,\\ &= -\bar{\psi}\gamma^0\gamma^2\gamma^5\gamma^{\mu\intercal}\gamma^0\gamma^2\psi,\\ &= \bar{\psi}\gamma^0\gamma^2\gamma^{\mu\intercal}\gamma^5\gamma^0\gamma^2\psi,\\ &= -\bar{\psi}\gamma^0\gamma^2\gamma^{\mu\intercal}\gamma^5\gamma^2\gamma^0\psi,\\ &= -\bar{\psi}\gamma^0\gamma^2\gamma^{\mu\intercal}\gamma^2\gamma^0\gamma^5\psi,\\ &= \bar{\psi}\gamma^\mu\gamma^5\psi = A^\mu.\end{aligned}$$

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c) We are to confirm the transformation properties of $P \equiv i\bar{\psi}\gamma^5\psi$ and $V^\mu \equiv \bar{\psi}\gamma^\mu\psi$ under $\mathcal{T}$.

First let us consider the transformation of the pseudo-scalar P.

$$\begin{aligned}\mathcal{T}P\mathcal{T}^\dagger = \mathcal{T}i\bar{\psi}\gamma^5\psi\mathcal{T}^\dagger &= -i(-\bar{\psi}\gamma^1\gamma^3)\gamma^5(\gamma^1\gamma^3\psi),\\ &= i\bar{\psi}\gamma^1\gamma^3\gamma^5\gamma^1\gamma^3\psi,\\ &= -i\bar{\psi}\gamma^5\psi = -P.\end{aligned}$$

Let us now consider the transformation of the vector V^μ.

$$\begin{aligned}\mathcal{T}V^\mu\mathcal{T}^\dagger = \mathcal{T}\bar{\psi}\gamma^\mu\psi\mathcal{T}^\dagger &= \bar{\psi}\gamma^3\gamma^1\gamma^{\mu *}\gamma^1\gamma^3\psi,\\ &= \bar{\psi}\gamma_\mu\psi = V_\mu.\end{aligned}$$

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2. a) We are to demonstrate the transformation properties of V^μ and A^μ, as previously defined, under $\mathcal{CP}$.

We have almost computed every detail necessary for our solution in question (1) above. The only transformation that we have not yet confirmed is the transformation of the vector V^μ under $\mathcal{P}$. Let us compute that now.

$$\begin{aligned}\mathcal{P}V^\mu\mathcal{P}^\dagger = \mathcal{P}\bar{\psi}\gamma^\mu\psi\mathcal{P}^\dagger &= \eta_a^*\bar{\psi}\gamma^0\gamma^\mu\eta_a\gamma^0\psi,\\ &= \bar{\psi}\gamma^0\gamma^\mu\gamma^0\psi,\\ &= \bar{\psi}\gamma_\mu\psi = V_\mu.\end{aligned}$$

By simply applying our transformation properties derived above in succession, we observe that,

$$\begin{aligned} V^\mu = \bar{\psi}\gamma^\mu\psi &\xrightarrow{\mathcal{P}} \bar{\psi}\gamma_\mu\psi \xrightarrow{\mathcal{C}} -\bar{\psi}\gamma_\mu\psi = -V_\mu\\ A^\mu = \bar{\psi}\gamma^\mu\gamma^0\psi &\xrightarrow{\mathcal{P}} -\bar{\psi}\gamma_\mu\gamma^5\psi \xrightarrow{\mathcal{C}} -\bar{\psi}\gamma_\mu\gamma^5\psi = -A_\mu\end{aligned}$$

b) Expecting an analogy with the electromagnetic current vector, we will check the transformation properties of each.

	agree?		agree?		agree?
$J^\mu \xrightarrow{\mathcal{P}} J_\mu$		$J^\mu \xrightarrow{\mathcal{C}} -J^\mu$		$J^\mu \xrightarrow{\mathcal{CP}} -J_\mu$	
$V^\mu \xrightarrow{\mathcal{P}} V_\mu$	*yes*	$V^\mu \xrightarrow{\mathcal{C}} -V^\mu$	*yes*	$V^\mu \xrightarrow{\mathcal{CP}} -V_\mu$	*yes*
$A^\mu \xrightarrow{\mathcal{P}} -A_\mu$	*no*	$A^\mu \xrightarrow{\mathcal{C}} A^\mu$	*no*	$A^\mu \xrightarrow{\mathcal{CP}} -A_\mu$	*yes*

c) We will demonstrate that the weak Lagrangian,

$$\mathcal{L}_{\text{weak}} \approx \frac{G_F}{\sqrt{2}}(V_\mu - A_\mu)(V^\mu - A^\mu),$$

is not invariant under $\mathcal{C}$ or $\mathcal{P}$, yet is invariant under $\mathcal{CP}$.

Like before, I will directly compute all of the transformations using the table made above in part (*b*) above. First note that

$$\mathcal{L}_{\text{weak}} \propto V^2 - 2V_\mu A^\mu + A^2.$$

When we take each of the of transformations from above, we see that

$$\begin{aligned} V^2 - 2V_\mu A^\mu + A^2 &\xrightarrow{\mathcal{P}} V^2 + 2V_\mu A^\mu + A^2 \neq \mathcal{L}_{\text{weak}};\\ V^2 - 2V_\mu A^\mu + A^2 &\xrightarrow{\mathcal{C}} V^2 + 2V_\mu A^\mu + A^2 \neq \mathcal{L}_{\text{weak}};\\ V^2 - 2V_\mu A^\mu + A^2 &\xrightarrow{\mathcal{CP}} V^2 - 2V_\mu A^\mu + A^2 = \mathcal{L}_{\text{weak}}.\end{aligned}$$

So $\mathcal{L}_{\text{weak}}$ is not invariant under $\mathcal{C}$ or $\mathcal{P}$ by is under $\mathcal{CP}$, as we were required to demonstrate.

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3. Let us define the product of the 3 discrete symmetry transformations as $\Theta \equiv \mathcal{CPT}$. We must show that under Θ, the Dirac field transforms by the rule

$$\Theta\psi(x)\Theta^\dagger = \gamma^5\psi^*(-x),$$

where

$$\psi^*(x) \equiv (\psi(x)^\dagger)^\intercal.$$

Like so many times before, we will proceed by direct calculation.

$$\begin{aligned}
\Theta\psi(x)\Theta^\dagger &= \mathcal{CPT}\psi(t,\vec{x})\mathcal{T}^\dagger\mathcal{P}^\dagger\mathcal{C}^\dagger,\\
&= \mathcal{CP}\gamma^1\gamma^3\psi(-t,\vec{x})\mathcal{P}^\dagger\mathcal{C}^\dagger,\\
&= \eta_a\mathcal{C}\gamma^1\gamma^3\gamma^0\psi(-x)\mathcal{C}^\dagger,\\
&= -i\eta_a\gamma^1\gamma^3\gamma^0(\psi(-x)^\dagger\gamma^2)^\intercal,\\
&= -i\eta_a\gamma^1\gamma^3\gamma^0\gamma^{2\intercal}\psi^*(-x),\\
&= -i\eta_a\gamma^0\gamma^1\gamma^3\gamma^2\psi^*(-x),\\
&= i\eta_a\gamma^0\gamma^1\gamma^2\gamma^3\psi^*(-x),\\
&= \eta_a\gamma^5\psi^*(-x).
\end{aligned}$$

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4. For the following derivations it will be useful to recall that

$$\gamma^0_W = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^i_W = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix},$$

where

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

a) We must show that any new matrices defined by

$$\gamma^\mu = U\gamma^\mu_W U^\dagger,$$

where U is an arbitrary 4×4 unitary matrix, satisfy the dirac algebra. This is proven by demonstrating that

$$\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}.$$

Knowing that the Weyl-representation γ^μ's satisfy the Dirac algebra, we will directly show that,

$$\begin{aligned}
\{\gamma^\mu,\gamma^\nu\} &= \{U\gamma^\mu_W U^\dagger, U\gamma^\nu_W U^\dagger\},\\
&= U\gamma^\mu_W U^\dagger U\gamma^\nu_W U^\dagger + U\gamma^\nu_W U^\dagger U\gamma^\mu_W U^\dagger,\\
&= U(\gamma^\mu_W\gamma^\nu_W + \gamma^\nu_W\gamma^\mu_W)U^\dagger,\\
&= 2Ug^{\mu\nu}U^\dagger = 2g^{\mu\nu}.
\end{aligned}$$

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b) Consider the unitary matrix which produces the Dirac representation

$$U_D = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

We must show that U_D is in fact unitary and we must find the matrices γ^μ in the Dirac representation.

The unitarity of U_D is trivial

$$U_D U_D^\dagger = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = 1_{4\times 4}.$$

When the matrices are directly computed, we see that

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^i = \gamma^i_W.$$

c) We must now show that in a general frame, the Dirac spinor takes the form,

$$u^s(p) = \begin{pmatrix} \sqrt{E+m}\xi^s \\ \vec{\sigma}\cdot\vec{p}\;\xi^s/\sqrt{E+m} \end{pmatrix}.$$

This is demonstrated by showing that it solves the Dirac equation, or, namely, that

$$\gamma^\mu p_\mu u^s(p) = m u^s(p).$$

This is simple to evaluate directly. Noting our Dirac representation of the γ^μ's and that $p_0 = E$, we see

$$\begin{aligned}
\gamma^\mu p_\mu u^s(p) &= \begin{pmatrix} p_0 & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -p_0 \end{pmatrix}\begin{pmatrix} \sqrt{E+m}\xi^s \\ \vec{\sigma}\cdot\vec{p}\;\xi^s/\sqrt{E+m} \end{pmatrix}, \\
&= \begin{pmatrix} \left[E\sqrt{E+m} - \frac{E^2-m^2}{\sqrt{E+m}}\right]\xi^s \\ \left[\vec{\sigma}\cdot\vec{p}\sqrt{E+m} - \frac{E\vec{\sigma}\cdot\vec{p}}{\sqrt{E+m}}\right]\xi^s \end{pmatrix}, \\
&= \begin{pmatrix} \sqrt{E+m}\left(E - \frac{E^2-m^2}{E+m}\right)\xi^s \\ \frac{\vec{\sigma}\cdot\vec{p}}{\sqrt{E+m}}(E+m-E)\xi^s \end{pmatrix}, \\
&= \begin{pmatrix} \sqrt{E+m}(E-E+m)\xi^s \\ m\vec{\sigma}\cdot\vec{p}\;\xi^s/\sqrt{E+m} \end{pmatrix}, \\
&= \begin{pmatrix} m\sqrt{E+m}\;\xi^s \\ m\vec{\sigma}\cdot\vec{p}\;\xi^s/\sqrt{E+m} \end{pmatrix}, \\
&= m u^s(p).
\end{aligned}$$

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d) We must show that the solution found in part (*c*) is normalized in the standard way.

Given that ξ is normalized such that $\xi\xi^\dagger = 1$, we see that

$$\begin{aligned}
\bar{u}u = u^\dagger\gamma^0 u &= \begin{pmatrix} \sqrt{E+m}\;\xi^\dagger & \vec{\sigma}\cdot\vec{p}\;\xi^\dagger/\sqrt{E+m} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \sqrt{E+m}\;\xi \\ \vec{\sigma}\cdot\vec{p}\;\xi/\sqrt{E+m} \end{pmatrix}, \\
&= \xi^\dagger\xi\left((E+m) - \frac{(\vec{\sigma}\cdot\vec{p})^2}{E+m}\right), \\
&= \frac{E^2+2mE+m^2-\vec{p}^2}{E+m}, \\
&= \frac{E^2+2mE+m^2-E^2+m^2}{E+m}, \\
&= \frac{2mE+2m^2}{E+m}, \\
&= 2m.
\end{aligned}$$

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Physics 513, Quantum Field Theory

Homework 7

Due Tuesday, 4^{th} November 2003

Jacob Lewis Bourjaily

Symmetry Factors

Throughout the following derivations it will be helpful to state explicitly a method to obtain the symmetry factor for a given diagram. The method is derived from the published lecture notes of Professor Colin Morningstar of Carnegie Mellon University.[1]

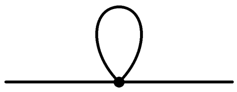
The *symmetry factor* of a given diagram is given by

$$S = \frac{n!(\eta)^n}{r},$$

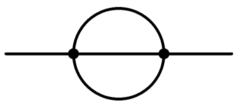
where n is number of vertices, η is a coupling constant, and r is the multiplicity of the diagram. The value of η is 4! in ϕ^4-theory and 3! in Yukawa theory. This pattern implies η will be 4 for question 1(b) below.

To determine the multiplicity r, all external points are labelled and all vertices are drawn with four (or three) lines emerging. All of these lines are assumed to be distinguishable. The total number of ways to connect the external points and vertices to form the diagram equals the multiplicity r. If a diagram is direction sensitive, then this is taken into account by only including the number of ways to draw the diagram given the directional conditions onf the external points.

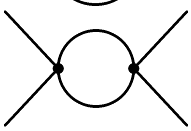
1. a) We are to determine the symmetry factor for four diagrams.



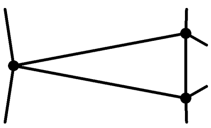
$r = 4 \cdot 3 = 12$ and $n = 1$ so $S = \frac{1!(4!)^1}{12} = 2$.



$r = 8 \cdot 4 \cdot 3 \cdot 2 = 192$ and $n = 2$ so $S = \frac{2!(4!)^2}{192} = 6$.

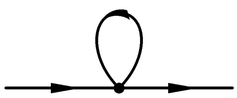


$r = 8 \cdot 3 \cdot 4 \cdot 3 \cdot 2 = 576$ and $n = 2$ so $S = \frac{2!(4!)^2}{576} = 2$.

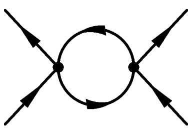


$r = 12 \cdot 3 \cdot 8 \cdot 3 \cdot 4 \cdot 3 \cdot 4 \cdot 2 = 82944$ and $n = 3$ so $S = \frac{3!(4!)^3}{82944} = 1$.

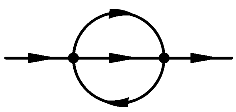
b) We are to determine the symmetry factors for the following diagrams.



$r = 2 \cdot 2 = 4$ and $n = 1$ so $S = \frac{1!(4)^1}{4} = 1$.



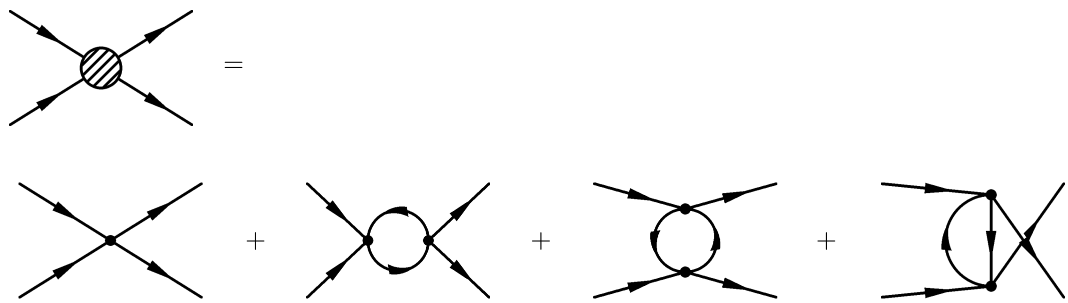
$r = 4 \cdot 2 \cdot 2 \cdot 2 = 32$ and $n = 2$ so $S = \frac{2!(4)^2}{32} = 1$.



$r = 4 \cdot 2 \cdot 2 = 16$ and $n = 2$ so $S = \frac{2!(4)^2}{16} = 2$.

[1]Chapter 9, page 141. Available at http://www.andrew.cmu.edu/course/33-770/.

2. a) We are asked to draw all distinct Feynman diagrams for the four point function of the ϕ^4-theory given below to the order λ^2.



i **ii** **iii** **iv**

b) We are to calculate the contributions from each diagram. Note that I have included explicit symmetry conservation in the above diagrams. For example, for contribution (ii), I have made the substitutions $k_1 = k$ and $k_2 = k - p_1 - p_2$; I have made similar substitutions for the other diagrams as well. Thus, including symmetry factors, the contributions are,

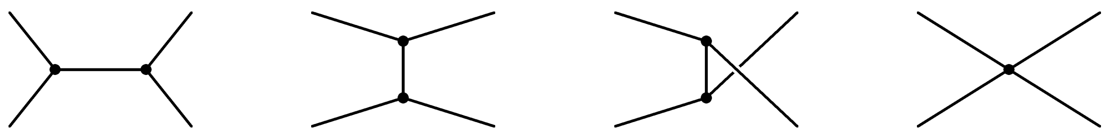
i) $(-i\lambda)(2\pi)^4\delta^{(4)}(p_3 + p_4 - p_1 - p_2)$;

ii) $\dfrac{(-i\lambda)^2}{2}\displaystyle\int\frac{d^4k}{(2\pi)^4}(2\pi)^4\delta^{(4)}(p_3+p_4-p_1-p_2)\frac{i}{(k^2-m^2+i\epsilon)}\frac{i}{((k-p_1-p_2)^2-m^2+i\epsilon)}$;

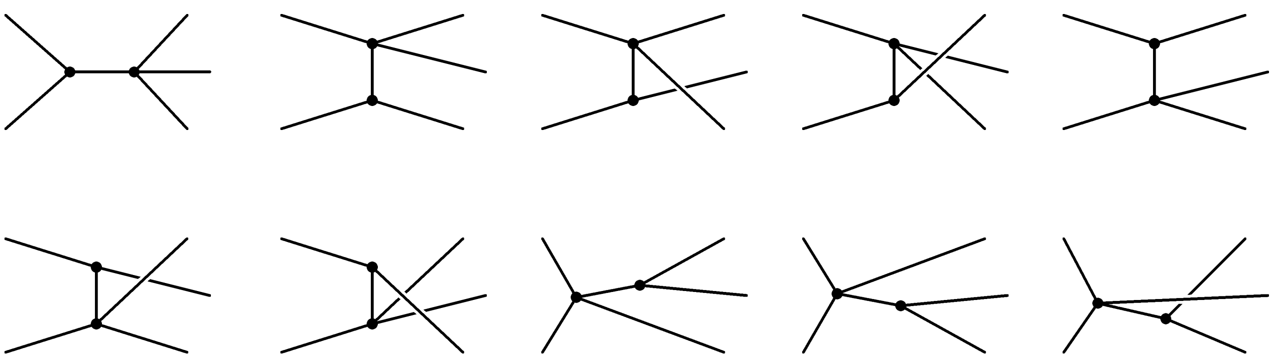
iii) $\dfrac{(-i\lambda)^2}{2}\displaystyle\int\frac{d^4k}{(2\pi)^4}(2\pi)^4\delta^{(4)}(p_3+p_4-p_1-p_2)\frac{i}{(k^2-m^2+i\epsilon)}\frac{i}{((k+p_1-p_3)^2-m^2+i\epsilon)}$;

iv) $\dfrac{(-i\lambda)^2}{2}\displaystyle\int\frac{d^4k}{(2\pi)^4}(2\pi)^4\delta^{(4)}(p_3+p_4-p_1-p_2)\frac{i}{(k^2-m^2+i\epsilon)}\frac{i}{((k+p_1-p_4)^2-m^2+i\epsilon)}$.

3. a) We are to draw all of the Feynman diagrams up to order λ or g^2 for the scattering process $p_A + p_B \to p_a + p_3$. These are given below.



b) Like part (a) above, we are to draw all of the Feynman diagrams of order $g\lambda$ for the process $p_A + p_B \to p_1 + p_2 + p_3$. Note that the labels are implied after the first diagram on the left of each row. There are 10, and they are given below.



c) It is clear that all of the symmetry factors are 1. I have directly computed them, but it is unnecessary to repeat those trivial calculations here. Rather, it is enough to notice that there are no loops in any of the diagrams. Each vertex connects unique, distinguishable fields. This is equivalent to the observation that the topology of each diagram above was enough to specify it entirely. Therefore, all symmetry factors are 1.

Physics 513, Quantum Field Theory

Homework 8

Due Tuesday, 11$^{\text{th}}$ November 2003

Jacob Lewis Bourjaily

Problem 4.1

We are to consider the problem of the creation of Klein Gordon particles by a classical source. This process can be described by the Hamiltonian

$$H = H_o + \int d^3x \ \ -j(x)\phi(x),$$

where H_o is the Klein-Gordon Hamiltonian, $\phi(x)$ is the Klein-Gordon filed, and $j(x)$ is a c-number scalar function. Let us define the number λ by the relation

$$\lambda = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}|\tilde{j}(p)|^2.$$

a) We are to show that the probability that the source creates *no* particles is given by

$$P(0) = \left|\langle 0|T\left\{\exp\left[i\int d^4x \ j(x)\phi_I(x)\right]\right\}|0\rangle\right|^2.$$

Without loss of understanding we will denote $\phi \equiv \phi_I$. Almost entirely trivially, we see that

$$H_I = -\int d^3x \ j(x)\phi(x).$$

Therefore,

$$\begin{aligned} P(0) &= \left|\langle 0|T\left\{\exp\left[-i\int dt' \ H_I(t')\right]\right\}|0\rangle\right|^2, \\ &= \left|\langle 0|T\left\{\exp\left[i\int d^4x \ j(x)\phi(x)\right]\right\}|0\rangle\right|^2. \end{aligned}$$

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b) We are to evaluate the expression for $P(0)$ to the order j^2 and show generally that $P(0) = 1-\lambda+\mathcal{O}(\lambda^2)$.

First, let us only consider the amplitude for the process. We can make the naïve expansion

$$\langle 0|T\left\{\exp\left[i\int d^4x \ j(x)\phi(x)\right]\right\}|0\rangle = \langle 0|1|0\rangle + i\int d^4x \ j(x)\langle 0|\phi(x)|0\rangle - \ldots.$$

For every odd power of the expansion, there will be at least one field ϕ_{I_o} that cannot be contracted from normal ordering and therefore will kill the entire term. So only even terms will contribute to the expansion. It should be clear that the amplitude will be of the form $\sim 1-\mathcal{O}(j^2)+\mathcal{O}(j^4)-\ldots$. Let us look at the $\mathcal{O}(j^2)$ term. That term is given by

$$\begin{aligned} \langle 0|T\left\{-\tfrac{1}{2}\left(\int d^4x \ j(x)\phi(x)\right)^2\right\}|0\rangle &= -\frac{1}{2}\int d^4xd^4y \ j(x)j(y)\langle 0|T\{\phi(x)\phi(y)\}|0\rangle, \\ &= -\frac{1}{2}\int d^4xd^4y \ j(x)j(y) \ D_F(x-y), \\ &= -\frac{1}{2}\int d^4xd^4y \ \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon}e^{-ip(x-y)}j(x)j(y), \\ &= -\frac{1}{2}\int \frac{d^4p}{(2\pi)^4} \ \underbrace{\int d^4x \ j(x)e^{-ipx}}_{\tilde{j}(p)}\underbrace{\int d^4y \ j(y)e^{ipy}}_{\tilde{j}^*(p)}\frac{i}{p^2-m^2+i\epsilon}, \\ &= -\frac{1}{2}\int \frac{d^4p}{(2\pi)^4}|\tilde{j}(p)|^2\frac{i}{p^2-m^2+i\epsilon}, \\ &= -\frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\int\frac{dp^0}{(2\pi)}|\tilde{j}(p)|^2\frac{i}{p^2-m^2+i\epsilon}. \end{aligned}$$

We know how to evaluate the integral

$$\int \frac{dp^0}{(2\pi)}|\tilde{j}\,(p)|^2 \frac{i}{p^2 - m^2 + i\epsilon} = \int \frac{dp^0}{(2\pi)}|\tilde{j}\,(p)|^2 \frac{i}{(p^0)^2 - E_{\mathbf{p}}^2 + i\epsilon},$$
$$= \int \frac{dp^0}{(2\pi)}|\tilde{j}\,(p)|^2 \frac{i}{(p^0 - E_{\mathbf{p}})(p^0 + E_{\mathbf{p}})}.$$

The function has a simple pole at $p^0 = -E_{\mathbf{p}}$ with the residue

$$\left.\frac{i|\tilde{j}\,(p)|^2}{p^0 - E_{\mathbf{p}}}\right|_{p^0 = -E_{\mathbf{p}}} = -\frac{i|\tilde{j}\,(p)|^2}{2E_{\mathbf{p}}}.$$

We know from elementary complex analysis that the contour integral is $2\pi i$ times the residue at the pole. Therefore,

$$-\frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\int \frac{dp^0}{(2\pi)}|\tilde{j}\,(p)|^2 \frac{i}{p^2 - m^2 + i\epsilon} = -\frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}|\tilde{j}\,(p)|^2,$$
$$= -\frac{1}{2}\lambda.$$

Because we now know the amplitude to the first order of λ (or, rather, the second order of j), we have shown, as desired, that

$$P(0) = |1 - \tfrac{1}{2}\lambda + \ldots|^2 \sim 1 - \lambda + \mathcal{O}(\lambda^2).$$

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c) We must represent the term computed in part (b) as a Feynman diagram and show that the whole perturbation series for $P(0)$ in terms of Feynman diagrams is precisely $P(0) = e^{-\lambda}$.

The term computed in part (b) can be represented by $\longrightarrow \equiv -\lambda$. It has two points (neither originated by the source) and a time direction specified (not to be confused with charge or momentum). We can write the entire perturbation series as

$$P(0) = \left|\langle 0|T\left\{\exp\left[i\int d^4x\; j(x)\phi(x)\right]\right\}|0\rangle\right|^2 = \left[1 + \longrightarrow + \begin{matrix}\longrightarrow\\ \longrightarrow\end{matrix} + \begin{matrix}\longrightarrow\\ \longrightarrow\\ \longrightarrow\end{matrix} + \begin{matrix}\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow\end{matrix} + \cdots\right]^2.$$

To get the series we must figure out the correct symmetry factors. If one begins with $2n$ vertices, then n of them must be chosen as 'in'; there are $2^{2n/2} = 2^n$ ways to do this. After that, each one of the 'in' vertices must be paired with one of the 'out' vertices; you can do this $n!$ ways. So the symmetry factor for the term with n uninteracting propagators is

$$S(n) = 2^n \cdot n!.$$

We may now compute the probability explicitly.

$$P(0) = \left[1 + \longrightarrow + \begin{matrix}\longrightarrow\\ \longrightarrow\end{matrix} + \begin{matrix}\longrightarrow\\ \longrightarrow\\ \longrightarrow\end{matrix} + \begin{matrix}\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow\end{matrix} + \cdots\right]^2,$$
$$= \left(\sum_{n=0}^{\infty}\frac{(-\lambda)^n}{2^n n!}\right)^2,$$
$$= \left(\sum_{n=0}^{\infty}\frac{(-\lambda/2)^n}{n!}\right)^2,$$
$$= \left(e^{-\lambda/2}\right)^2,$$
$$\therefore P(0) = e^{-\lambda}.$$

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d) Let us now compute the probability that the source creates one particle of momentum k. First we should perform this computation to $\mathcal{O}(j)$ and then to all orders using the same trick as in part (c) to sum the series.

Let us calculate the amplitude that a particle is created with the explicit momentum $\mathbf{k}$.

$$\begin{aligned}
&\langle 0|T\left\{\phi_k \exp\left[i\int d^4x\ j(x)\phi(x)\right]\right\}|0\rangle \\
&= i\int d^4x\ j(x)\langle 0|a_{\mathbf{k}}\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(a_{\mathbf{p}}e^{-ipx} + a_{\mathbf{p}}^\dagger e^{ipx}\right)|0\rangle|0\rangle, \\
&= i\int d^4x\ j(x)\langle 0|a_{\mathbf{k}}\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}a_{\mathbf{p}}^\dagger e^{ipx}|0\rangle, \\
&= i\int d^4x\ j(x)\langle 0|\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}e^{ipx}(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{k})|0\rangle, \\
&= i\int \frac{d^4x}{(2\pi)^4}\frac{j(x)}{\sqrt{2E_{\mathbf{k}}}}e^{ikx}, \\
&= \frac{i\tilde{j}(k)}{\sqrt{2E_{\mathbf{k}}}}.
\end{aligned}$$

Now, the probability of creating such a particle is the modulus of the amplitude.

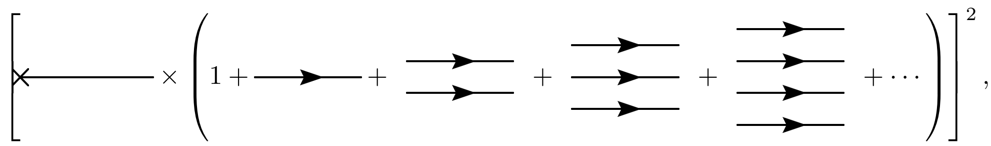
$$P(1_{\mathbf{k}}) = \frac{|\tilde{j}(x)|^2}{2E_{\mathbf{k}}}.$$

We can compute the probability that a particle is created with any momentum by simply integrating over all the possible $\mathbf{k}$. This yields

$$P(1) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2E_{\mathbf{k}}}|\tilde{j}(x)|^2 = \lambda.$$

Therefore in Feynman graphs, $\equiv i\sqrt{\lambda}$. The entire perturbation in Feynman diagrams is therefore

$$P(1) =$$



$$= \left|i\sqrt{\lambda}e^{\lambda/2}\right|^2,$$

$$\therefore P(1) = \lambda e^{-\lambda}.$$

e) We are to show that the probability of producing n particles is given by a Poisson distribution.

From part (d) above, we know that each creation vertex on the Feynman diagram must be multiplied by $i\sqrt{\lambda}$. Now, because each of the final products are identical and there are $n!$ ways of arranging them, the symmetry factor in each case is $n!$. The probability is approximated by

$$P(n) \sim \frac{\lambda^n}{n!}.$$

Like we have done before, to get the correct probability, we must take into account the probability that no particle is created. Therefore,

$$P(n) = \frac{\lambda^n e^{-\lambda}}{n!}.$$

f) We must show that a poisson distribution given above with parameter λ has a norm of 1, an expectation value of λ, and a variance of λ.

First, let us compute the norm of the distribution function.

$$\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}e^{\lambda} = e^{-\lambda}\sum_{n=0}^{\infty}\frac{\lambda^n}{n!} = e^{-\lambda}e^{\lambda} = 1.$$

The expectation value for the number created is simply,

$$E(n) = \sum_{n=0}^{\infty} \frac{n\lambda^n}{n!} e^{-\lambda} = \lambda e^{-\lambda} \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{(n-1)!} = \lambda e^{-\lambda} \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} = \lambda e^{-\lambda} e^{\lambda} = \lambda.$$

To compute the variance, we will use the relation $Var(n) = E(n^2) - E(n)^2$. Let us compute $E(n^2)$.

$$\begin{aligned}
E(n^2) &= \sum_{k=0}^{\infty} n^2 \frac{\lambda^n}{n!} e^{-\lambda}, \\
&= \lambda e^{-\lambda} \sum_{n=1}^{\infty} n \frac{\lambda^{n-1}}{(n-1)!}, \\
&= \lambda e^{-\lambda} \sum_{n=1}^{\infty} ((n-1)+1) \frac{\lambda^{n-1}}{(n-1)!}, \\
&= \lambda e^{-\lambda} \left[\sum_{n=1}^{\infty} (n-1) \frac{\lambda^{n-1}}{(n-1)!} + \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{(n-1)!} \right], \\
&= \lambda e^{-\lambda} e^{\lambda} + \lambda e^{-\lambda} \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{(n-2)!}, \\
&= \lambda^{+} \lambda^2 e^{-\lambda} \sum_{n=2}^{\infty} \frac{\lambda^{n-2}}{(n-2)!}, \\
&= \lambda^2 + \lambda.
\end{aligned}$$

Knowing this, it is clear that

$$Var(n) = \lambda^2 + \lambda - \lambda = \lambda.$$

Problem 4.4

The cross section for scattering of an electron by the Coulomb field of a nucleus can be computed, to lowest order, without quantizing the electromagnetic field. We will treat the field as a given. classical potential $A_\mu(x)$. The interaction Hamiltonian is then

$$H_I = \int d^3x \; e\bar{\psi}\gamma^\mu \psi A_\mu,$$

where $\psi(x)$ is the usual quantized Dirac field.

a) We must show that the T-matrix element for an electron scatter to off a localized classical potential is given to the lowest order by

$$\langle p_f | iT | p_i \rangle = -ie\bar{u}(p_f)\gamma^\mu u(p_i) \cdot \tilde{A}_\mu(p_f - p_i).$$

where $\tilde{A}_\mu$ is the Fourier transform of A_μ.

We may compute this contribution directly.

$$\begin{aligned}
\langle p_f | iT | p \rangle &= -i \int d^4x \langle p_f | T\{H_I(x)\} | p_i \rangle, \\
&= -ie \int d^4x \; A_\mu \langle p_f | T\{\bar{\psi}(x)\gamma^\mu \psi(x)\} | p_i \rangle, \\
&= -ie \int d^4x \; A_\mu \langle p_f | \bar{\psi}(x)\gamma^\mu \psi(x) | p_i \rangle, \\
&= -ie \int d^4x \; A_\mu(x) \overline{u}^{s'}(p_f)\gamma^\mu u^s(p_i) e^{ix(p_f - p_i)}, \\
&= -ie\overline{u}^{s'}(p_f)\gamma^\mu u^s(p_i) \int d^4x \; A_\mu(x) e^{ix(p_f - p_i)}, \\
&= -ie\overline{u}^{s'}(p_f)\gamma^\mu u^s(p_i) \tilde{A}_\mu(p_f - p_i).
\end{aligned}$$

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b) If $A_\mu(x)$ is time independent, its Fourier transform contains a delta function of energy. We therefore define

$$\langle p_f|iT|p_i\rangle \equiv i\mathcal{M}\cdot(2\pi)\delta(E_f-E_i).$$

Given this definition of $\mathcal{M}$, we must show that the cross section for scattering off a time-independent localized potential is given by

$$d\sigma = \frac{1}{v_i}\frac{1}{2E_i}\frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}(2\pi)\delta(E_f-E_i)|\mathcal{M}(p_i\to p_f)|^2.$$

From class we know that we can represent an incoming wave packet with momentum p_i in the z-direction and impact parameter b by the relation

$$|\psi_b\rangle = \int\frac{d^3p_i}{(2\pi)^3}\frac{1}{\sqrt{2E_{p_i}}}e^{-ibp_i}\psi(p_i)|p_i\rangle.$$

The probability of interaction given an impact parameter is then

$$\begin{aligned}
P(b) &= \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}|\langle p_f|iT|\psi_b\rangle|^2,\\
&= \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\int\frac{d^3p_id^3k}{(2\pi)^6}\frac{1}{\sqrt{2E_{p_i}2E_k}}e^{-ib(p_i-k)}\psi(p_i)\psi^*(k)\langle p_f|iT|p_i\rangle\langle p_f|iT|k\rangle^*,\\
&= \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\int\frac{d^3p_id^3k}{(2\pi)^6}\frac{e^{-ib(p_i-k)}}{\sqrt{2E_{p_i}2E_k}}\psi(p_i)\psi^*(k)(2\pi)^2\delta(E_f-E_{p_i})\delta(E_f-E_k)\mathcal{M}(p_i\to p_f)\mathcal{M}(k\to p_f)^*.
\end{aligned}$$

Therefore,

$$\begin{aligned}
d\sigma &= \int d^2b\ P(b),\\
&= \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\int d^2b\frac{d^3pd^3k}{(2\pi)^6}\frac{e^{-ib(p-k)}}{\sqrt{2E_p2E_k}}\psi(p)\psi^*(k)(2\pi)^2\delta(E_f-E_p)\delta(E_f-E_k)\mathcal{M}(p\to p_f)\mathcal{M}(k\to p_f)^*,\\
&= \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\int\frac{d^3pd^3k}{(2\pi)^6}\frac{\psi(p)\psi^*(k)}{\sqrt{2E_p2E_k}}(2\pi)^2\delta^{(2)}(p_\perp-k_\perp)\delta(E_f-E_p)\delta(E_f-E_k)\mathcal{M}(p\to p_f)\mathcal{M}(k\to p_f)^*,\\
&= \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\frac{1}{|v_i|}(2\pi)\int\frac{d^3pd^3k}{(2\pi)^3}\frac{\psi(p)\psi^*(k)}{\sqrt{2E_p2E_k}}\delta^{(2)}(p_\perp-k_\perp)\delta(p_z-k_z)\delta(E_f-E_p)\mathcal{M}(p\to p_f)\mathcal{M}(k\to p_f)^*,\\
&= \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\frac{1}{|v_i|}(2\pi)\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}|\psi(p)|^2\delta(E_f-E_p)|\mathcal{M}(p\to p_f)|^2,
\end{aligned}$$

With a properly normalized wave function, this reduces directly to (allow me to apologize for the inconsistency with notation. It is hard to keep track of. The incoming momentum p has energy E_i.)

$$d\sigma = \frac{1}{v_i}\frac{1}{2E_i}\frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}(2\pi)\delta(E_f-E_i)|\mathcal{M}(p_i\to p_f)|^2.$$

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Now, let us try to write an expression for $d\sigma/d\Omega$.

$$\begin{aligned}
\int d\sigma &= \int\frac{d^3p_f}{(2\pi)^3}\frac{1}{v_i}\frac{1}{2E_i}\frac{1}{2E_f}(2\pi)\delta(E_f-E_i)|\mathcal{M}|^2,\\
&= \int\frac{p_f^2dp_fd\Omega}{(2\pi)^2}\frac{1}{v_i}\frac{1}{2E_f2E_i}\frac{1}{v_f}\delta(p'-p)|\mathcal{M}|^2,\\
&= \int\frac{d\Omega}{(2\pi)^2}\frac{p^2}{4v_i^2E_i^2}|\mathcal{M}|^2,\\
&= \int d\Omega\frac{1}{16\pi^2}|\mathcal{M}|^2.
\end{aligned}$$

Therefore, we have that

$$\frac{d\sigma}{d\Omega} = \frac{1}{16\pi^2}|\mathcal{M}|^2.$$

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c) We will now specialize to the non-relativistic scattering of a Coulomb potential ($A^0 = Ze/4\pi r$). We must show that in this limit

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2 Z^2}{4m^2 v^4 \sin^4(\theta/2)}.$$

Let us first take the Fourier transform of the Coulomb potential.

$$\tilde{A}_\mu(\mathbf{k}) = \frac{Ze}{4\pi}\int d^3r \frac{e^{i\mathbf{kr}}}{\mathbf{r}},$$
$$= \frac{Ze}{4\pi}\frac{4\pi}{\mathbf{k}^2},$$
$$\therefore \tilde{A}_\mu(\mathbf{k}) = \frac{Ze}{\mathbf{k}^2}.$$

From part (a) above, we calculated that

$$\mathcal{M} = -ie\overline{u}^{s'}(p_f)\gamma^\mu u^s(p)\tilde{A}_\mu(p_f - p),$$
$$= \frac{-ie^2 Z}{(p_f - p)^2}\overline{u}^{s'}(p_f)\gamma^0 u^s(p).$$

In the nonrelativistic limit, $E >> p$ so we may approximate that

$$\overline{u}^{s'}(p_f)\gamma^0 u^s(p) = u^{s'\dagger}(p_f)u^s(p) = 2E\delta^{s's}.$$

Therefore, our amplitude becomes

$$\mathcal{M} = \frac{-ie^2 Z}{(p_f - p)^2}2E\delta^{s's}.$$

From part (b), we may compute $d\sigma/d\Omega$ directly.

$$\frac{d\sigma}{d\Omega} = \frac{4Z^2 e^4 E62}{16\pi^2(p_f - p)^4},$$
$$= \frac{Z^2\alpha^2 E^2}{p^4(1 - cos\theta)^2},$$
$$= \frac{Z^2\alpha^2 E^2}{4p^4\sin^4(\theta/2)},$$
$$= \frac{Z^2\alpha^2}{4E^2 v^4\sin^4(\theta/2)}.$$

In the nonrelativistic limit, we have that $E^2 \sim m^2$. Therefore we may conclude as desired that

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2 Z^2}{4m^2 v^4 \sin^4(\theta/2)}.$$

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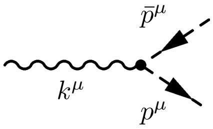
Physics 513, Quantum Field Theory
Homework 9
Due Tuesday, 18th November 2003

Jacob Lewis Bourjaily

The Decay of Vector into Two Scalars

We are to compute the decay rate of unpolarized vector particles of mass M into two scalars of mass m. We should calculate the decay rate in the rest frame.

Defining $\tilde{p}^\mu = (\bar{p} - p)^\mu$, the amplitude for the decay diagram is given by



$$= i\mathcal{M} = \epsilon_\mu i f \tilde{p}^\mu.$$

It is quite straightforward to calculate the spin-averaged square of the amplitude,

$$\begin{aligned}\overline{|\mathcal{M}|^2} &= \frac{1}{3}\sum_{\text{spin}} \epsilon_\mu i f \tilde{p}^\mu \epsilon_\nu^* (-i) f \tilde{p}^\nu, \\ &= \frac{f^2}{3}\left(\frac{k_\mu k_\nu}{M^2} - g_{\mu\nu}\right)\tilde{p}^\mu\tilde{p}^\nu, \\ &= \frac{f^2}{3}\left(\frac{(k_\mu \tilde{p}^\mu)^2}{M^2} - \tilde{p}^2\right).\end{aligned}$$

Now, because we are computing this in the rest frame where $k_\mu = (M, 0)$ and $\tilde{p}^\mu = (0, -2|\vec{p}|)$, $k_\mu \tilde{p}^\mu = 0$. Similarly, we know that $\tilde{p}^2 = 4|\vec{p}|^2$. Therefore,

$$\overline{|\mathcal{M}|^2} = \frac{4f^2|\vec{p}|^2}{3}.$$

Note that $|\vec{p}| = E^2 - m^2 = \left(\frac{M^2}{4} - m^2\right)^{1/2}$. Using this and the equation for the decay rate found in Peskin and Schroeder,

$$\begin{aligned}\Gamma &= \frac{1}{2M}\int \frac{d\Omega}{16\pi^2}\frac{|\vec{p}|}{M}|\mathcal{M}|^2, \\ &= \frac{1}{2M}\int \frac{d\Omega}{16\pi^2}\frac{|\vec{p}|}{M}\frac{4f^2|\vec{p}|^2}{3}, \\ &= \frac{f^2}{24\pi^2 M^2}\int d\Omega |\vec{p}|^3,\end{aligned}$$

$$\therefore \Gamma = \frac{f^2\left(\frac{M^2}{4} - m^2\right)^{3/2}}{6\pi M^2}.$$

Mott's Formula

We are to generalize problem 2 of Homework 8 in the relativistic case. We computed then the general amplitude to be

$$\mathcal{M} = \frac{-ie^2 Z}{(p_f - p)^2}\bar{u}^{s'}(p_f)\gamma^0 u^s(p).$$

To compute the spin averaged amplitude, it will be helpful to recall our earlier kinematic result that $(p_f - p)^4 = 16|\vec{p}|^4 \sin^4\theta/2$. Let us now compute the amplitude squared in the spin-averaged case.

$$\begin{aligned}\overline{|\mathcal{M}|^2} &= \frac{1}{2}\frac{Z^2 e^4}{(p_f - p)^4}\sum_{\text{spin}} \bar{u}^s(p)\gamma^0 u^{s'}(p_f)\bar{u}^{s'}(p_f)\gamma^0 u^s(p), \\ &= \frac{Z^2 e^4}{32|\vec{p}|^4 \sin^4\theta/2}\text{Tr}\left(\gamma^0(\not{p}_f + m)\gamma^0(\not{p} + m)\right).\end{aligned}$$

It will be helpful to break up the trace into its four additive pieces.

$$\text{Tr}\left(\gamma^0(\not{p}_f + m)\gamma^0(\not{p} + m)\right) = \text{Tr}\left(\gamma^0\not{p}_f\gamma^0\not{p}\right) + \text{Tr}\left(\gamma^0 m\gamma^0\not{p}\right) + \text{Tr}\left(\gamma^0\not{p}_f\gamma^0 m\right) + \text{Tr}\left(\gamma^0 m\gamma^0 m\right).$$

It should be clear that the two middle terms are both zero because there is an odd number of γ's. The last term is nearly trivial, $\mathrm{Tr}\left(\gamma^0 m\gamma^0 m\right) = 4m^2$. Let us now work on the first term.

$$\begin{aligned}
\mathrm{Tr}\left(\gamma^0 \not{p}_f \gamma^0 \not{p}\right) &= p_{f_\mu} p_\nu \mathrm{Tr}\left(\gamma^0\gamma^\mu\gamma^0\gamma^\nu\right),\\
&= 4p_{f_\mu}p_\nu\left(g^{0\mu}g^{0\nu} - g^{00}g^{\mu\nu} + g^{0\nu}g^{\mu 0}\right),\\
&= 4\left(2E^2 - p_{f_\mu}p^\mu\right),\\
&= 4\left(2E^2 - E^2 + \vec{p_f}\vec{p}\right),\\
&= 4\left(E^2 + |\vec{p}|^2\cos\theta\right).
\end{aligned}$$

Using these results, we have that

$$\begin{aligned}
\overline{|\mathcal{M}|^2} &= \frac{Z^2e^4}{8|\vec{p}|^4\sin^4\theta/2}\left[E^2 + |\vec{p}|^2\cos\theta + m^2\right],\\
&= \frac{Z^2e^4}{8|\vec{p}|^4\sin^4\theta/2}\left[2E^2 - |\vec{p}|^2(1-\cos\theta)\right],\\
&= \frac{Z^2e^4}{8|\vec{p}|^4\sin^4\theta/2}\left[2E^2 - 2|\vec{p}|^2\sin^2\theta/2\right],\\
&= \frac{Z^2e^4E^2}{4|\vec{p}|^4\sin^4\theta/2}\left[1 - \left(\frac{|\vec{p}|}{E}\right)^2\sin^2\theta/2\right],\\
&= \frac{Z^2e^4}{4\beta^2|\vec{p}|^2\sin^4\theta/2}\left[1 - \beta^2\sin^2\theta/2\right].
\end{aligned}$$

In the last two lines we have used the fact that $\vec{p}/E = \beta$. Now, we showed in Homework 8 that

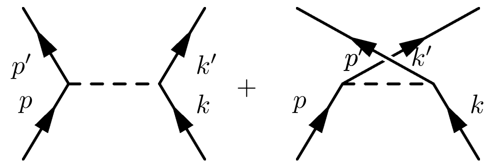
$$\frac{d\sigma}{d\Omega} = \frac{|\mathcal{M}|^2}{16\pi^2}.$$

Using the fine structure constant to simplify notation, where $\alpha^2 = \frac{e^4}{16\pi^2}$, it is clear that

$$\therefore \frac{d\sigma}{d\Omega} = \frac{Z^2\alpha^2}{4\beta^2|\vec{p}|^2\sin^4\theta/2}\left[1 - \beta^2\sin^2\theta/2\right].$$

Helicity Amplitudes in Yukawa Theory

We are to consider the amplitude given by,

$i\mathcal{M} =$ 

$$= (-ig^2)\left(\bar{u}(p')u(p)\frac{1}{(p'-p)^2 - m_\phi^2}\bar{u}(k')u(k) - \bar{u}(p')u(k)\frac{1}{(p'-k)^2 - m_\phi^2}\bar{u}(k')u(p)\right).$$

a) We are to derive the selection rules for helicity for this theory.

We can best understand the selection rules by requiring that one of the spinors is in a projection. To bring the projection operator to the neighboring spinor (in either diagram and starting from any outside term) requires that the projection anticommutes through a γ^0. Therefore, the interaction *must* flip the spins. Exempli Gratia, $\bar{u}\frac{1+\gamma^5}{2}u_R = u^\dagger\gamma^0\frac{1+\gamma^5}{2}u_R = \bar{u}_L u_R$.

b) Given these selection rules, what are the non-vanishing amplitudes? These are the only possible terms that involve both incoming states flipping their spin in the outgoing states. So, the nonzero amplitudes are $\mathcal{M}_{LL;RR}, \mathcal{M}_{RR;LL}, \mathcal{M}_{LR;RL}, \mathcal{M}_{RL;LR}, \mathcal{M}_{RL;RL}, \mathcal{M}_{LR;LR}$.

c) We are to use problem 5 of Homework 5 to compute the explicit form of the two-spinors. We should use this to find the eigenvectors $u_\lambda(p)$ at very high energies. This is a relatively straight forward calculation. We derived quite some time ago that in the high energy limit for general spinors. Using the helicity basis derived in Homework 5, we see that

$$u_R = \sqrt{2E}\begin{pmatrix} 0 \\ 0 \\ -e^{-i\phi}\sin\theta/2 \\ \cos\theta/2 \end{pmatrix} \qquad \text{and} \qquad u_L = \sqrt{2E}\begin{pmatrix} \cos\theta/2 \\ e^{i\phi}\sin\theta/2 \\ 0 \\ 0 \end{pmatrix}.$$

d) Now we should rederive the selection rules from part (a). This is relatively straight forward. Let us compute directly the $\bar{u}_R$ and $\bar{u}_L$ These two are simply,

$$\bar{u}_R = \left(-\sqrt{2E}e^{i\phi}\sin\theta/2, \sqrt{2E}\cos\theta/2, 0, 0\right) \qquad \text{and} \qquad \bar{u}_L = \left(0, 0, \sqrt{2E}\cos\theta/2, \sqrt{2E}e^{-i\phi}\sin\theta/2\right).$$

It should be clear that in this limit, $\bar{u}_R u_R = 0$ because the have opposite zeros. Therefore, we may again conclude that the only inner products that do not vanish are those which flip the spin at the vertex. This is the same relationship seen intuitively in part (a).

e) We must now compute the nonvanishing inner produces of the eigenvectors that we mentioned above. Let us compute each in turn directly.

$$\begin{aligned}
\bar{u}_R(p')u_L(p) &= -2Ee^{i\phi}\sin\theta/2; \\
\bar{u}_L(p')u_R(p) &= 2Ee^{-i\phi}\sin\theta/2; \\
\bar{u}_R(k')u_L(p) &= -2Ee^{i\phi}\cos\theta/2; \\
\bar{u}_L(k')u_R(p) &= 2Ee^{-i\phi}\cos\theta/2; \\
\bar{u}_R(p')u_L(k) &= 2Ee^{i\phi}\cos\theta/2; \\
\bar{u}_L(p')u_R(k) &= -2Ee^{-i\phi}\cos\theta/2; \\
\bar{u}_R(k')u_L(k) &= 2Ee^{i\phi}\sin\theta/2; \\
\bar{u}_L(k')u_R(k) &= -2Ee^{-i\phi}\sin\theta/2.
\end{aligned}$$

f) Let us compute the amplitudes $\mathcal{M}_{RR;LL}$ and $\mathcal{M}_{LR;LR}$ in the limit of very high energy. We use the limit to reduce $|\vec{p}|^2$-like terms to E^2. These are directly computed to be

$$\begin{aligned}
\mathcal{M}_{RR;LL} &= -g^2\left((-2Ee^{i\phi}\sin\theta/2)\frac{1}{4E^2\sin^2\theta/2}2Ee^{i\phi}\sin\theta/2 - 2Ee^{i\phi}\cos\theta/2\frac{1}{-4E^2\cos^2\theta/2}(-2Ee^{i\phi}\cos\theta/2)\right), \\
&= g^2(e^{i\phi} + e^{i\phi}),
\end{aligned}$$

$$\therefore \mathcal{M}_{RR;LL} = 2g^2e^{i\phi}.$$

By a similar calculation,

$$\mathcal{M}_{LR;LR} = -g^2\left(2Ee^{-i\phi}\cos\theta/2\frac{1}{-4E^2\cos^2\theta/2}(-2Ee^{i\phi}\cos\theta/2)\right),$$

$$\therefore \mathcal{M}_{LR;LR} = -g^2.$$

g) Let us determine the spin averaged amplitude squared. The contributions are very similar to the two above (in fact, the amplitudes are identical so we just multiply). We see

$$\overline{|\mathcal{M}|^2} = \frac{1}{4}\left(2(2g^2)^2 + 4g^4\right),$$

$$\therefore \overline{|\mathcal{M}|^2} = 3g^4.$$

Physics 513, Quantum Field Theory

Homework 10

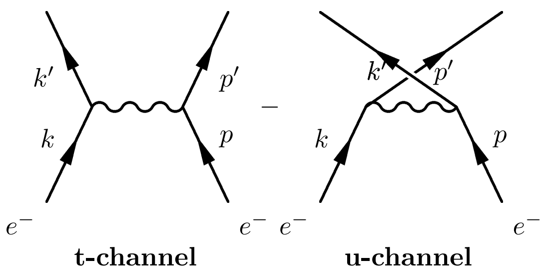
Due Tuesday, 25th November 2003

Jacob Lewis Bourjaily

Electron-Electron Scattering We are to consider the elastic scattering of two electrons (M/øller scattering) in Quantum Electrodynamics.

a) We are to draw the two tree-level Feynman diagrams for the scattering amplitude. We see that they are,

$i\mathcal{M}_{\lambda'\mu';\lambda\mu} =$



$$= ie^2\left[\bar{u}_{\mu'}(k')\gamma^\mu u_\mu(k)\frac{1}{(k-k')^2}\bar{u}_{\lambda'}(p')\gamma_\mu u_\lambda(p) - \bar{u}_{\mu'}(k')\gamma^\mu u_\lambda(p)\frac{1}{(p-k')^2}\bar{u}_{\lambda'}(p')\gamma_\mu u_\mu(k)\right].$$

The relative minus sign is a simple consequence of Fermi statistics.

b) Using the Gordon identity, derived in homework 5 problem 3, we are to derive a simple form of the amplitude for the forward most direction. Here we will assume that $p' \sim p$. So,

$$\begin{aligned}
\bar{u}_{\lambda'}(p')\gamma^\mu u_{\lambda'}(p) &= \bar{u}_{\lambda'}(p')\left[\frac{(p'+p)^\mu}{2m} + \frac{i\sigma^{\mu\nu}(p'-p)_\nu}{2m}\right]u_\lambda(p),\\
&= \bar{u}_{\lambda'}(p)\left[\frac{(p+p)^\mu}{2m} + \frac{i\sigma^{\mu\nu}(p-p)_\nu}{2m}\right]u_\lambda(p),\\
&= \bar{u}_{\lambda'}(p)\frac{p^\mu}{m}u_\lambda(p),\\
&= 2p^\mu\delta_{\lambda'\lambda}.
\end{aligned}$$

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c) In the forward most direction, it is clear that the denominator for the t-channel contribution is small and the denominator for the u-channel contribution is large so only the t-channel contributions are relevant. In the t-channel amplitude, it is clear that spin cannot flip so that spin of the initial and final particles are the same. Therefore, the important terms are $\mathcal{M}_{LL;LL}, \mathcal{M}_{RR;RR}, \mathcal{M}_{RL;RL}, \mathcal{M}_{LR;LR}$.

d) In contrast to part (c), only the u-channel contributions are important so the final spin states may be switched. So the important amplitudes are $\mathcal{M}_{LL;LL}, \mathcal{M}_{RR;RR}, \mathcal{M}_{LR;RL}, \mathcal{M}_{RL;LR}$.

e) Using parts (a) and (b), we may compute,

$$\begin{aligned}
\mathcal{M}_{PR;LR} &= e^2 \bar{u}_{\mu'}(k)\gamma^\mu u_\mu(k)\frac{1}{(k-k')^2}\bar{u}_{\lambda'}(p)\gamma_\mu u_\lambda(p),\\
&= \frac{e^2}{-2\vec{k}^2(1-\cos\theta)}4k^\mu p_\mu,\\
&= \frac{e^2}{-\vec{k}^2\sin^2\theta/2}k^\mu p_\mu,\\
&= \frac{e^2}{-\vec{k}^2\sin^2\theta/2}\left(\frac{E_{cm}^2}{4}+\vec{k}^2\right),\\
&= \frac{e^2\left(\frac{E_{cm}^2}{2}-m^2\right)}{-\left(\frac{E_{cm}^2}{4}-m^2\right)\sin^2\theta/2},\\
&= -\frac{2e^2\left(E_{cm}^2-2m^2\right)}{(E_{cm}^2-4m^2)\sin^2\theta/2}.
\end{aligned}$$

f) We now should compute the differential cross section with respect to the scattering angle θ.

$$\begin{aligned}
\frac{d\sigma}{d\cos\theta} &= 2\pi\frac{d\sigma}{d\Omega},\\
&= \frac{2\pi|\mathcal{M}|^2}{64\pi^2E_{cm}^2},\\
&= \frac{2\pi\alpha^2 4\left(E_{cm}^2-2m^2\right)^2}{4E_{cm}^2\left(E_{cm}^2-4m^2\right)^2\sin^4\theta/2},\\
\therefore \frac{d\sigma}{d\cos\theta} &= \frac{2\pi\alpha^2\left(E_{cm}^2-2m^2\right)^2}{E_{cm}^2\left(E_{cm}^2-4m^2\right)^2\sin^4\theta/2}.
\end{aligned}$$

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A Delicate Balance Consider the reactions $a+b\to a'+b'$ and $a'+b'\to a+b$. These four particles may all have different masses and different spins given by $s_a, s_b, s_{a'}, s_{b'}$. We are to compute the ratio of differential cross sections with respect to solid angle Ω for the two processes.

Because of the enormous symmetry of the two processes, it will suffice to demonstrate a calculation of one of the processes. We will assume the process is time reversal so that the amplitude squared is the same for both. Let us compute the differential cross section.

$$\begin{aligned}
\frac{d\sigma}{d\Omega}(a+b\to a'+b') &= \frac{\vec{p}}{2E_a2E_b|v_a-v_b|(2\pi)^24E_{cm}}\frac{|\mathcal{M}(\text{sum})|^2}{(2s_a+1)(2s_b+1)},\\
&= \frac{\vec{p}|\mathcal{M}(\text{sum})|^2}{64\pi^2E_aE_b\vec{p}(1/E_a+1/E_b)E_{cm}(2s_a+1)(2s_b+1)},\\
&= \frac{|\mathcal{M}(\text{sum})|^2}{64\pi^2E_aE_b(1/E_a+1/E_b)(E_a+E_b)(2s_a+1)(2s_b+1)},\\
&= \frac{|\mathcal{M}(\text{sum})|^2}{64\pi^2(E_a+E_b)^2(2s_a+1)(2s_b+1)},\\
&= \frac{|\mathcal{M}(\text{sum})|^2}{64\pi^2k^2(2s_a+1)(2s_b+1)}.
\end{aligned}$$

Now, it is clear by symmetry that this implies

$$\frac{d\sigma}{d\Omega}(a'+b'\to a+b) = \frac{|\mathcal{M}(\text{sum})|^2}{64\pi^2k'^2(2s_{a'}+1)(2s_{b'}+1)}.$$

$$\therefore \frac{\frac{d\sigma}{d\Omega}(a+b\to a'+b')}{\frac{d\sigma}{d\Omega}(a'+b'\to a+b)} = \frac{k'^2(2s_{a'}+1)(2s_{b'}+1)}{k^2(2s_a+1)(2s_b+1)}.$$

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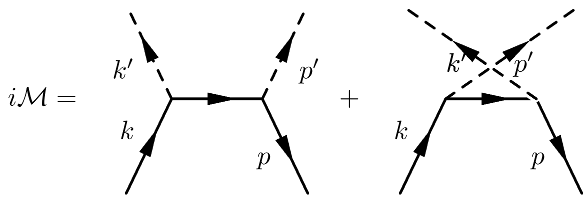
Fermion Annihilation in Yukawa Theory We are to consider the process of fermion anti-fermion annihilation into to scalars $f\bar{f} \to \phi\phi$.

a,b) The two Feynman diagrams for the S-matrix in the tree approximation are,



t-channel **u-channel**

$$= (-ig^2)\left[u^2(p)\frac{\not{p} - \not{p}' + m}{(p-p')^2 - m^2}\bar{v}^r(k) + u^s(p)\frac{\not{p} - \not{k}' + m}{(p-k')^2 - m^2}\bar{v}^r(k)\right].$$

c) The relative sign is because of Bose statistics.

Physics 513, Quantum Field Theory

Homework 11

Due Thursday, 4th December 2003

Jacob Lewis Bourjaily

The Dirac Propagator

a) The Dirac propagator is defined as the time-ordered two point correlator

$$S_F(x-y)_{ab} = \langle 0|T\{\psi_a(x)\bar{\psi}_b(y)\}|0\rangle = \begin{cases} \langle 0|\psi_a(x)\bar{\psi}_b(y)|0\rangle & x^0 > y^0 \\ -\langle 0|\bar{\psi}_b(y)\psi_a(x)|0\rangle & y^0 > x^0 \end{cases}.$$

We are to evaluate $S_F(x-y)_{ab}$ for a free Dirac field ψ.

Let us first compute this for the case when $x^0 > y^0$; the other case will follow trivially from symmetry arguments. Dropping all obviously zero terms, we may immediately write that

$$S_F(x-y)_{ab} = \langle 0|\int \frac{d^3p}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{p}'}}}e^{i(p'y-px)}\sum_{\text{spin}}\left(a^s_{\mathbf{p}}a^{s'\dagger}_{\mathbf{p}'}u^s_a(p)\bar{u}^{s'}_b(p')\right)|0\rangle.$$

Now, we know that $\langle 0|a^s_{\mathbf{p}}a^{s'\dagger}_{\mathbf{p}'}|0\rangle = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}')\delta_{ss'}$ so

$$S_F(x-y)_{ab} = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{-ip(x-y)}\sum_{\text{spin}}u^s_a(p)\bar{u}^s_b(p),$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}(\not{p}+m)_{ab}e^{-ip(x-y)},$$

$$\therefore S_F(x-y)_{ab} = (i\not{\partial}+m)_{ab}D(x-y) \quad |x^0 > y^0.$$

Now, we see that when $y^0 > x^0$, the propagator will involve the sum over spins of the v spinors which will give a $-(i\not{\partial}+m)$. This minus is cancelled by the minus in the definition of the two-point correlator.

$$\therefore S_F(x-y)_{ab} = (i\not{\partial}+m)_{ab}D(y-x) \quad |y^0 > x^0.$$

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b) We are to show that the Dirac propagator is a Green's function. Let us write the propagator as

$$S_F(x-y)_{ab} = \theta(x^0-y^0)\langle 0|\psi_a(x)\bar{\psi}_b(y)|0\rangle - \theta(y^0-x^0)\langle 0|\bar{\psi}_b(y)\psi_a(x)|0\rangle.$$

When we act on this with $(i\not{\partial}-m)$, it is clear that much of the mess that follows can be greatly simplified by simple considerations. First, note that by the chain rule we will have to have terms where the partial acts on the Heaviside function multiplied by the correlator together with terms where the Heaviside function is multiplied by the partial acting on the correlator. The $-m$ term will come through the Heaviside functions and the net effect will be to have terms similar to $(i\not{\partial}-m)\langle 0|\psi_a(x)\bar{\psi}_b(y)|0\rangle$ which will be identically zero by Dirac's equations. The only terms left will be the partial derivatives acting on the Heaviside functions. This can be further simplified because $\partial\theta(x^0-y^0) = -\partial\theta(y^0-x^0)$. Therefore the entire operation reduces to

$$(i\not{\partial}-m)S_F(x-y)_{ab} = i\gamma^0\delta(x^0-y^0)\langle 0|\{\psi_a(x),\bar{\psi}_b(y)\}|0\rangle,$$

$$= i\gamma^0\delta(x^0-y^0)\langle 0|\{\psi_a(x),\psi^\dagger_b(y)\gamma^0\}|0\rangle,$$

$$= i(\gamma^0)^2\delta(x^0-y^0)\{\psi_a(x),\psi^\dagger_b(y)\}\langle 0|0\rangle,$$

$$= i\delta(x^0-y^0)d^{(3)}(\vec{x}-\vec{y})\delta_{ab},$$

$$\therefore (i\not{\partial}-m)S_F(x-y)_{ab} = i\delta^{(4)}(x-y)\delta_{ab}.$$

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c) We are to solve the equation for the Green's function equation by introducing the Fourier transform
$$S_C(x-y)_{ab} = \int_C \frac{d^4p}{(2\pi)^4}\tilde{S}_C(p)_{ab}e^{-ip(x-y)},$$
and express our answer in terms of the scalar propagator
$$D_C(x-y) = \int_C \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2}e^{-ip(x-y)}.$$

This can be done in a rather straight-forward way. We will write the Green's function equation of part (b) in terms of the prescribed substitution for $S_C(x-y)_{ab}$.
$$(\not{p}-m)\int_C \frac{d^4p}{(2\pi)^4}\tilde{S}_C(p)_{ab}e^{-ip(x-y)} = i\delta^{(4)}(x-y).$$
We can of course bring $(\not{p}-m)$ inside the integral and it is clear that the only way for this identity to be true is if
$$(\not{p}-m)\tilde{S}_C(p)_{ab} = i.$$
If this is the case than the exponential will reduce to a simple Dirac delta functional multiplied by i which is precisely what we want. So
$$\begin{aligned}\tilde{S}_C(p) &= \frac{i}{(\not{p}-m)},\\ &= \frac{i}{(\not{p}-m)}\frac{(\not{p}+m)}{(\not{p}+m)},\\ &= \frac{i(\not{p}+m)}{p^2-m^2},\\ \therefore S_C(x-y)_{ab} = \int_C \frac{d^4p}{(2\pi)^4}\frac{i(\not{p}+m)}{p^2-m^2}e^{-ip(x-y)} &= (\not{p}+m)D_C(x-y).\end{aligned}$$

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d) We are to use the identity
$$D_F(x-y) = \theta(x^0-y^0)D(x-y)+\theta(y^0-x^0)D(y-x),$$
together with the relation derived in part (c) to reproduce the results of part (a).

Let us first write out our explicit formulation of the Dirac propagator.
$$S_F(x-y) = (i\not{\partial}+m)\left(\theta(x^0-y^0)D(x-y)+\theta(y^0-x^0)D(y-x)\right).$$

Like before, we will use argumentation to reduce the problem rather than writing out explicit terms. When we act with the partial derivative operator on the Heaviside functions, we get a relative minus sign between the two terms and they will exactly cancel. They did not cancel in part (b) because there was already an inherent minus sign between the two terms. Now, because they begin additive, they will cancel. The net effect will be to bring our entire operator $(i\not{\partial}+m)$ inside the Heaviside functions completely. This will result in
$$S_F(x-y) = \theta(x^0-y^0)(i\not{\partial}+m)D(x-y)+\theta(y^0-x^0)(i\not{\partial}+m)D(y-x).$$

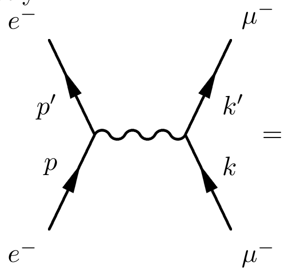
If you look at the two derived terms in part (a) they are identical to the equation above. Therefore, we nearly trivially reproduce the results of part (a).

Mott's Formula (II)

In homework 9, we derived Mott's formula (the relativistic Rutherford formula). We are now to derive it by considering the spin-averaged amplitude squared of the scattering of an electron with a muon in the limit that the mass of the muon is much larger than the energy of the electron.

a) We are to compute the spin-averaged amplitude squared for $e^-\mu^-$ scattering for general m_e and m_μ.

Let us compute this directly.

$i\mathcal{M} =$ 

$= \frac{ie^2}{q^2}\bar{u}(p')\gamma^\mu u(p)\bar{u}(k')\gamma_\mu u(k).$

We can compute the spin-averaged square of the amplitude directly. This becomes

$$\begin{aligned}
\overline{|\mathcal{M}|^2} &= \frac{e^4}{4q^4}\mathrm{Tr}\left[(\not{p}' + m_e)\gamma^\mu(\not{p} + m_e)\gamma^\nu\right]\mathrm{Tr}\left[(\not{k}' + m_\mu)\gamma_\mu(\not{k} + m_\mu)\gamma_\nu\right], \\
&= \frac{4e^4}{q^4}\left[p'^\mu p^\nu + p'^\nu p^\mu + g\mu\nu(m_e^2 - p\cdot p')\right] \times \left[k'_\mu k_\nu + k'_\nu k_\mu + g_{\mu\nu}(m_\mu^2 - k\cdot k')\right], \\
\therefore \overline{|\mathcal{M}|^2} &= \frac{8e^4}{q^4}\left[(p\cdot k') + (p'\cdot k){}_{(}p\cdot k)(p'\cdot k') - m_\mu^2(p\cdot p') - m_e^2(k\cdot k') + 2m_\mu^2 m_e^2\right].
\end{aligned}$$

b) Taking the limit where m_μ is large, we can consider the case that the center of mass frame of the collision is the muon's rest frame. Therefore, we have that $k \approx k' = (m_\mu, \vec{0})$. E represents the energy of the electron. In this case, we can drastically simplify our kinematics.

$$p\cdot k = Em_\mu \qquad k\cdot k' = m_\mu^2 \qquad p\cdot p' = E^2 - \vec{p}\vec{p'} = E^2 - \vec{p}^{\,2}\cos\theta.$$

We can use this to directly write our spin-averaged squared amplitude

$$\begin{aligned}
\overline{|\mathcal{M}|^2} &= \frac{8e^4}{q^4}\left[2E^2m_\mu^2 - m_\mu^2(E^2 - \vec{p}^{\,2}\cos\theta) + m_e^2 m_\mu^2\right], \\
&= m_\mu^2\frac{16e^4}{q^4}\left(E^2 - \vec{p}^{\,2}\sin^2\theta/2\right), \\
\therefore \overline{|\mathcal{M}|^2} &= \frac{m_\mu^2 e^4}{\beta^2\vec{p}^{\,2}\sin^4\theta/2}\left(1 - \beta^2\sin^2\theta/2\right).
\end{aligned}$$

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In the last step we reduced the formula to one which will greatly help us in part (c) below.

c) We are to derive Mott's formula by taking the limit where m_μ is very large in the center of mass frame. As we stated before, this approximation is identical to assuming that the center of mass frame is actually the rest frame of the muon so our amplitude calculated in part (b) is correct to the second order. We know that the final velocity of the muon is zero and that the center of mass energy is approximately m_μ (to the first order) in this frame so we may write,

$$\begin{aligned}
\left.\frac{d\sigma}{d\Omega}\right|_{\text{cm}} &= \frac{1}{4E_aE_b|v_a - v_b|}\frac{|\vec{p}|}{(2\pi)^2 4E_{cm}}\overline{|\mathcal{M}|^2}, \\
&= \frac{1}{4Em_\mu\beta}\frac{|\vec{p}|}{(2\pi)^2 4m_\mu}\frac{m_\mu^2 e^4}{\beta^2\vec{p}^{\,2}\sin^4\theta/2}\left(1 - \beta^2\sin^2\theta/2\right), \\
&= \frac{e^4}{16\pi^2 4\beta^2\vec{p}^{\,2}\sin^4\theta/2}\left(1 - \beta^2\sin^2\theta/2\right), \\
\therefore \left.\frac{d\sigma}{d\Omega}\right|_{\text{cm}} &= \frac{\alpha^2}{4\beta^2\vec{p}^{\,2}\sin^4\theta/2}\left(1 - \beta^2\sin^2\theta/2\right).
\end{aligned}$$

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Physics 513, Quantum Field Theory
Examination 1

Due Tuesday, 28^{th} October 2003

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1. a) We are to verify that in the Schödinger picture we may write the total momentum operator,

$$\mathbf{P} = -\int d^3x\ \pi(\mathbf{x})\nabla\phi(\mathbf{x}),$$

in terms of ladder operators as

$$\mathbf{P} = \int \frac{d^3p}{(2\pi)^3}\ \mathbf{p}\ a^\dagger_{\mathbf{p}} a_{\mathbf{p}}.$$

Recall that in the Schrödinger picture, we have the following expansions for the fields ϕ and π in terms of the bosonic ladder operators

$$\phi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3}\ \frac{1}{\sqrt{2E_{\mathbf{p}}}} e^{i\mathbf{px}} \left(a_{\mathbf{p}} + a^\dagger_{-\mathbf{p}}\right); \tag{1.1}$$

$$\pi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3}\ (-i)\sqrt{\frac{E_{\mathbf{p}}}{2}} e^{i\mathbf{px}} \left(a_{\mathbf{p}} - a^\dagger_{-\mathbf{p}}\right). \tag{1.2}$$

To begin our derivation, let us compute $\vec{\nabla}\phi(\mathbf{x})$.

$$\begin{aligned}
\nabla\phi(\mathbf{x}) &= \nabla\int \frac{d^3p}{(2\pi)^3}\ \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left(a_{\mathbf{p}} e^{i\mathbf{px}} + a^\dagger_{\mathbf{p}} e^{-i\mathbf{px}}\right),\\
&= \int \frac{d^3p}{(2\pi)^3}\ \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left(i\mathbf{p} a_{\mathbf{p}} e^{i\mathbf{px}} - i\mathbf{p} a^\dagger_{\mathbf{p}} e^{-i\mathbf{px}}\right),\\
&= \int \frac{d^3p}{(2\pi)^3}\ \frac{1}{\sqrt{2E_{\mathbf{p}}}} i\mathbf{p} e^{i\mathbf{px}} \left(a_{\mathbf{p}} - a^\dagger_{-\mathbf{p}}\right).
\end{aligned}$$

Using this and (1.2) we may write the expression for $\mathbf{P}$ directly.

$$\begin{aligned}
\mathbf{P} &= -\int d^3x\ \pi(\mathbf{x})\nabla\phi(\mathbf{x}),\\
&= -\int d^3x \frac{d^3k d^3p}{(2\pi)^6}\ \frac{1}{2}\sqrt{\frac{E_{\mathbf{k}}}{E_{\mathbf{p}}}} \mathbf{p} e^{i(\mathbf{p}+\mathbf{k})\mathbf{x}} \left(a_{\mathbf{k}} - a^\dagger_{-\mathbf{k}}\right)\left(a_{\mathbf{p}} - a^\dagger_{-\mathbf{p}}\right),\\
&= \int \frac{d^3k d^3p}{(2\pi)^6}\ \frac{-1}{2}\sqrt{\frac{E_{\mathbf{k}}}{E_{\mathbf{p}}}} \mathbf{p} (2\pi)^3 \delta^{(3)}(\mathbf{p}+\mathbf{k}) \left(a_{\mathbf{k}} - a^\dagger_{-\mathbf{k}}\right)\left(a_{\mathbf{p}} - a^\dagger_{-\mathbf{p}}\right),\\
&= \int \frac{d^3p}{(2\pi)^3}\ \frac{1}{2}\mathbf{p} \left(a^\dagger_{\mathbf{p}} - a_{-\mathbf{p}}\right)\left(a_{\mathbf{p}} - a^\dagger_{-\mathbf{p}}\right).
\end{aligned}$$

Using symmetry we may show that $a_{-\mathbf{p}} a^\dagger_{-\mathbf{p}} = a_{\mathbf{p}} a^\dagger_{\mathbf{p}}$. With this, our total momentum becomes,

$$\mathbf{P} = \int \frac{d^3p}{(2\pi)^3}\ \frac{1}{2}\mathbf{p} \left(a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + a_{\mathbf{p}} a^\dagger_{\mathbf{p}}\right).$$

By adding and then subtracting $a^\dagger_{\mathbf{p}} a_{\mathbf{p}}$ inside the parenthesis, one sees that

$$\begin{aligned}
\mathbf{P} &= \int \frac{d^3p}{(2\pi)^3}\ \frac{1}{2}\mathbf{p} \left(2a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + [a_{\mathbf{p}}, a^\dagger_{\mathbf{p}}]\right),\\
&= \int \frac{d^3p}{(2\pi)^3} \mathbf{p} \left(a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + \tfrac{1}{2}\, [a_{\mathbf{p}}, a^\dagger_{\mathbf{p}}]\right).
\end{aligned}$$

Unfortunately, we have precisely the same problem that we had with the Hamiltonian: there is an infinite 'baseline' momentum. Of course, our 'justification' here will be identical to the one offered in that case and so

$$\boxed{\therefore \mathbf{P} = \int \frac{d^3p}{(2\pi)^3}\ \mathbf{p}\ a^\dagger_{\mathbf{p}} a_{\mathbf{p}}.} \tag{1.3}$$

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b) We are to verify that the Dirac charge operator,

$$Q = \int d^3x \ \psi^\dagger(x)\psi(x),$$

may be written in terms of ladder operators as

$$Q = \int \frac{d^3p}{(2\pi)^3} \sum_s \left(a_{\mathbf{p}}^{s\dagger} a_{\mathbf{p}}^s - b_{\mathbf{p}}^{s\dagger} b_{\mathbf{p}}^s\right).$$

Recall that we can expand our Dirac ψ's in terms of fermionic ladder operators.

$$\psi_a(x) = \int \frac{d^3p}{(2\pi)^3} \ \frac{1}{\sqrt{2E_{\mathbf{p}}}} e^{i\mathbf{px}} \sum_s \Big(a_{\mathbf{p}}^s u_a^s(\mathbf{p}) + b_{-\mathbf{p}}^{s\dagger} v_a^s(-\mathbf{p})\Big); \tag{1.4}$$

$$\psi_b(x)^\dagger = \int \frac{d^3p}{(2\pi)^3} \ \frac{1}{\sqrt{2E_{\mathbf{p}}}} e^{-i\mathbf{px}} \sum_r \Big(a_{\mathbf{p}}^{r\dagger} u_b^{r\dagger}(\mathbf{p}) + b_{-\mathbf{p}}^{r} v_b^{r\dagger}(-\mathbf{p})\Big). \tag{1.5}$$

Therefore, we can compute Q by writing out its terms explicitly.

$$\begin{aligned}
Q &= \int d^3x \ \psi^\dagger(x)\psi(x), \\
&= \int d^3x \frac{d^3k d^3p}{(2\pi)^6} \frac{1}{2\sqrt{E_{\mathbf{k}} E_{\mathbf{p}}}} e^{i(\mathbf{p}-\mathbf{k})\mathbf{x}} \sum_{r,s} \Big[\Big(a_{\mathbf{k}}^{r\dagger} u_b^{r\dagger}(\mathbf{k}) + b_{-\mathbf{k}}^{r} v_b^{r\dagger}(-\mathbf{k})\Big)\Big(a_{\mathbf{p}}^s u_a^s(\mathbf{p}) + b_{-\mathbf{p}}^{s\dagger} v_a^s(-\mathbf{p})\Big)\Big], \\
&= \int \frac{d^3k d^3p}{(2\pi)^6} \frac{1}{2\sqrt{E_{\mathbf{k}} E_{\mathbf{p}}}} (2\pi)^3 \delta^{(3)}(\mathbf{p}-\mathbf{k}) \sum_{r,s} \Big[\Big(a_{\mathbf{k}}^{r\dagger} u_b^{r\dagger}(\mathbf{k}) + b_{-\mathbf{k}}^{r} v_b^{r\dagger}(-\mathbf{k})\Big)\Big(a_{\mathbf{p}}^s u_a^s(\mathbf{p}) + b_{-\mathbf{p}}^{s\dagger} v_a^s(-\mathbf{p})\Big)\Big], \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \sum_{r,s} \Big[\Big(a_{\mathbf{p}}^{r\dagger} u_b^{r\dagger}(\mathbf{p}) + b_{-\mathbf{p}}^{r} v_b^{r\dagger}(-\mathbf{p})\Big)\Big(a_{\mathbf{p}}^s u_a^s(\mathbf{p}) + b_{-\mathbf{p}}^{s\dagger} v_a^s(-\mathbf{p})\Big)\Big], \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \sum_{r,s} \Big(a_{\mathbf{p}}^{r\dagger} a_{\mathbf{p}}^s u_b^{r\dagger}(\mathbf{p}) u_a^s(\mathbf{p}) + a_{\mathbf{p}}^{r\dagger} b_{-\mathbf{p}}^{s\dagger} u_b^{r\dagger}(\mathbf{p}) v_a^s(-\mathbf{p}) \\
&\qquad\qquad + b_{-\mathbf{p}}^{r} a_{\mathbf{p}}^s v_b^{r\dagger}(-\mathbf{p}) u_a^s(\mathbf{p}) + b_{-\mathbf{p}}^{r} b_{-\mathbf{p}}^{s\dagger} v_b^{r\dagger}(-\mathbf{p}) v_a^s(-\mathbf{p})\Big), \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \sum_{r,s} \Big(a_{\mathbf{p}}^{r\dagger} a_{\mathbf{p}}^s u_b^{r\dagger}(\mathbf{p}) u_a^s(\mathbf{p}) + b_{-\mathbf{p}}^{r} b_{-\mathbf{p}}^{s\dagger} v_b^{r\dagger}(-\mathbf{p}) v_a^s(-\mathbf{p})\Big), \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} 2E_{\mathbf{p}} \delta^{rs} \sum_{r,s} \Big(a_{\mathbf{p}}^{r\dagger} a_{\mathbf{p}}^s + b_{-\mathbf{p}}^{r} b_{-\mathbf{p}}^{s\dagger}\Big), \\
&= \int \frac{d^3p}{(2\pi)^3} \sum_s \Big(a_{\mathbf{p}}^{s\dagger} a_{\mathbf{p}}^s + b_{-\mathbf{p}}^{s} b_{-\mathbf{p}}^{s\dagger}\Big),
\end{aligned}$$

We note that by symmetry $b_{-\mathbf{p}}^s b_{-\mathbf{p}}^{s\dagger} = b_{\mathbf{p}}^s b_{\mathbf{p}}^{s\dagger}$. By using its anticommutation relation to rewrite $b_{\mathbf{p}}^s b_{\mathbf{p}}^{s\dagger}$ and then dropping the infinite 'baseline' energy as we did in part (a), we see that

$$\boxed{\therefore Q = \int \frac{d^3p}{(2\pi)^3} \sum_s \left(a_{\mathbf{p}}^{s\dagger} a_{\mathbf{p}}^s - b_{\mathbf{p}}^{s\dagger} b_{\mathbf{p}}^s\right).} \tag{1.6}$$

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2. a) We are to show that the matrices

$$(\mathcal{J}^{\mu\nu})_{\alpha\beta} = i\left(\delta^{\mu}{}_{\alpha}\delta^{\nu}{}_{\beta} - \delta^{\mu}{}_{\beta}\delta^{\nu}{}_{\alpha}\right),$$

generate the Lorentz algebra,

$$[\mathcal{J}^{\mu\nu}, \mathcal{J}^{\rho\sigma}] = i\left(g^{\nu\rho}\mathcal{J}^{\mu\sigma} - g^{\mu\rho}\mathcal{J}^{\nu\sigma} - g^{\nu\sigma}\mathcal{J}^{\mu\rho} + g^{\mu\sigma}\mathcal{J}^{\nu\rho}\right).$$

We are reminded that matrix multiplication is given by $(AB)_{\alpha\gamma} = A_{\alpha\beta}B^{\beta}{}_{\gamma}$. Recall that in homework 5.1, we showed that

$$(\mathcal{J}^{\mu\nu})^{\alpha}{}_{\beta} = i\left(g^{\mu\alpha}\delta^{\nu}{}_{\beta} - g^{\nu\alpha}\delta^{\mu}{}_{\beta}\right).$$

Let us proceed directly to demonstrate the Lorentz algebra.

$$\begin{aligned}
[\mathcal{J}^{\mu\nu}, \mathcal{J}^{\rho\sigma}] &= -\left(\delta^{\mu}{}_{\alpha}\delta^{\nu}{}_{\beta} - \delta^{\mu}{}_{\beta}\delta^{\nu}{}_{\alpha}\right)\left(g^{\rho\beta}\delta^{\sigma}{}_{\gamma} - g^{\sigma\beta}\delta^{\rho}{}_{\gamma}\right) + \left(\delta^{\rho}{}_{\alpha}\delta^{\sigma}{}_{\beta} - \delta^{\rho}{}_{\beta}\delta^{\sigma}{}_{\alpha}\right)\left(g^{\mu\beta}\delta^{\nu}{}_{\gamma} - g^{\nu\beta}\delta^{\mu}{}_{\gamma}\right),\\
&= -\underbrace{\delta^{\mu}{}_{\alpha}\delta^{\nu}{}_{\beta}g^{\rho\beta}\delta^{\sigma}{}_{\gamma}}_{1} + \underbrace{\delta^{\mu}{}_{\alpha}\delta^{\nu}{}_{\beta}g^{\sigma\beta}\delta^{\rho}{}_{\gamma}}_{2} + \underbrace{\delta^{\mu}{}_{\beta}\delta^{\nu}{}_{\alpha}g^{\rho\beta}\delta^{\sigma}{}_{\gamma}}_{3} - \underbrace{\delta^{\mu}{}_{\beta}\delta^{\nu}{}_{\alpha}g^{\sigma\beta}\delta^{\rho}{}_{\gamma}}_{4}\\
&\qquad + \underbrace{\delta^{\rho}{}_{\alpha}\delta^{\sigma}{}_{\beta}g^{\mu\beta}\delta^{\nu}{}_{\gamma}}_{5} - \underbrace{\delta^{\rho}{}_{\alpha}\delta^{\sigma}{}_{\beta}g^{\nu\beta}\delta^{\mu}{}_{\gamma}}_{6} - \underbrace{\delta^{\rho}{}_{\beta}\delta^{\sigma}{}_{\alpha}g^{\mu\beta}\delta^{\nu}{}_{\gamma}}_{7} + \underbrace{\delta^{\rho}{}_{\beta}\delta^{\sigma}{}_{\alpha}g^{\nu\beta}\delta^{\mu}{}_{\gamma}}_{8},\\
&= -(\underbrace{\delta^{\mu}{}_{\alpha}\delta^{\nu}{}_{\beta}g^{\rho\beta}\delta^{\sigma}{}_{\gamma} - \delta^{\rho}{}_{\beta}\delta^{\sigma}{}_{\alpha}g^{\nu\beta}\delta^{\mu}{}_{\gamma}}_{1\&8})g^{\nu\rho} + (\underbrace{\delta^{\mu}{}_{\alpha}\delta^{\nu}{}_{\beta}g^{\sigma\beta}\delta^{\rho}{}_{\gamma} - \delta^{\rho}{}_{\alpha}\delta^{\sigma}{}_{\beta}g^{\nu\beta}\delta^{\mu}{}_{\gamma}}_{2\&6})g^{\nu\sigma}\\
&\qquad + (\underbrace{\delta^{\mu}{}_{\beta}\delta^{\nu}{}_{\alpha}g^{\rho\beta}\delta^{\sigma}{}_{\gamma} - \delta^{\rho}{}_{\beta}\delta^{\sigma}{}_{\alpha}g^{\mu\beta}\delta^{\nu}{}_{\gamma}}_{3\&7})g^{\mu\rho} - (\underbrace{\delta^{\mu}{}_{\beta}\delta^{\nu}{}_{\alpha}g^{\sigma\beta}\delta^{\rho}{}_{\gamma} - \delta^{\rho}{}_{\alpha}\delta^{\sigma}{}_{\beta}g^{\mu\beta}\delta^{\nu}{}_{\gamma}}_{4\&5})g^{\mu\sigma},
\end{aligned}$$

$$\boxed{\therefore [\mathcal{J}^{\mu\nu}, \mathcal{J}^{\rho\sigma}] = i\left(g^{\nu\rho}\mathcal{J}^{\mu\sigma} - g^{\mu\rho}\mathcal{J}^{\nu\sigma} - g^{\nu\sigma}\mathcal{J}^{\mu\rho} + g^{\mu\sigma}\mathcal{J}^{\nu\rho}\right).} \tag{2.1}$$

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b) Like part (a) above, we are to show that the matrices

$$S^{\mu\nu} = \frac{i}{4}[\gamma^{\mu}, \gamma^{\nu}],$$

generate the Lorentz algebra,

$$[S^{\mu\nu}, S^{\rho\sigma}] = i\left(g^{\nu\rho}S^{\mu\sigma} - g^{\mu\rho}S^{\nu\sigma} - g^{\nu\sigma}S^{\mu\rho} + g^{\mu\sigma}S^{\nu\rho}\right).$$

As Pascal wrote, 'I apologize for the length of this [proof], for I did not have time to make it short.' Before we proceed directly, let's outline the derivation so that the algebra is clear. First, we will fully expand the commutator of $S^{\mu\nu}$ with $S^{\rho\sigma}$. We will have 8 terms. For each of those terms, we will use the anticommutation identity $\gamma^{\mu}\gamma^{\nu} = 2g^{\mu\nu} - \gamma^{\nu}\gamma^{\mu}$ to rewrite the middle of each term. By repeated use of the anticommutation relations, it can be shown that

$$\gamma^{\mu}\gamma^{\rho}\gamma^{\nu}\gamma^{\sigma} = \gamma^{\sigma}\gamma^{\nu}\gamma^{\rho}\gamma^{\mu} + 2(g^{\nu\sigma}\gamma^{\mu}\gamma^{\rho} - g^{\rho\sigma}\gamma^{\mu}\gamma^{\nu} + g^{\mu\sigma}\gamma^{\rho}\gamma^{\nu} - g^{\rho\nu}\gamma^{\sigma}\gamma^{\mu} + g^{\mu\nu}\gamma^{\sigma}\gamma^{\rho} - g^{\mu\rho}\gamma^{\sigma}\gamma^{\nu}), \tag{2.2}$$

This will be used to cancel many terms and multiply the whole expression by 2 before we contract back to terms involving $S^{\mu\nu}$'s. Let us begin.

$$\begin{aligned}
[S^{\mu\nu}, S^{\rho\sigma}] &= -\frac{1}{16}\left([\gamma^{\mu}\gamma^{\nu} - \gamma^{\nu}\gamma^{\mu}, \gamma^{\rho}\gamma^{\sigma} - \gamma^{\sigma}\gamma^{\rho}]\right),\\
&= -\frac{1}{16}\left([\gamma^{\mu}\gamma^{\nu}, \gamma^{\rho}\gamma^{\sigma}] - [\gamma^{\mu}\gamma^{\nu}, \gamma^{\sigma}\gamma^{\rho}] - [\gamma^{\nu}\gamma^{\mu}, \gamma^{\rho}\gamma^{\sigma}] + [\gamma^{\nu}\gamma^{\mu}, \gamma^{\sigma}\gamma^{\rho}]\right),\\
&= -\frac{1}{16}\left(\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma} - \gamma^{\rho}\gamma^{\sigma}\gamma^{\mu}\gamma^{\nu} - \gamma^{\mu}\gamma^{\nu}\gamma^{\sigma}\gamma^{\rho} + \gamma^{\sigma}\gamma^{\rho}\gamma^{\mu}\gamma^{\nu}\right.\\
&\qquad\left. -\gamma^{\nu}\gamma^{\mu}\gamma^{\rho}\gamma^{\sigma} + \gamma^{\rho}\gamma^{\sigma}\gamma^{\nu}\gamma^{\mu} + \gamma^{\nu}\gamma^{\mu}\gamma^{\sigma}\gamma^{\rho} - \gamma^{\sigma}\gamma^{\rho}\gamma^{\nu}\gamma^{\mu}\right),\\
&= -\frac{1}{16}\Big(2g^{\nu\rho}\gamma^{\mu}\gamma^{\sigma} - \gamma^{\mu}\gamma^{\rho}\gamma^{\nu}\gamma^{\sigma} - 2g^{\nu\rho}\gamma^{\sigma}\gamma^{\mu} + \gamma^{\sigma}\gamma^{\nu}\gamma^{\rho}\gamma^{\mu}\\
&\qquad - 2g^{\sigma\mu}\gamma^{\rho}\gamma^{\nu} + \gamma^{\rho}\gamma^{\mu}\gamma^{\sigma}\gamma^{\nu} + 2g^{\sigma\mu}\gamma^{\nu}\gamma^{\rho} - \gamma^{\nu}\gamma^{\sigma}\gamma^{\mu}\gamma^{\rho}\\
&\qquad - 2g^{\nu\sigma}\gamma^{\mu}\gamma^{\rho} + \gamma^{\mu}\gamma^{\sigma}\gamma^{\nu}\gamma^{\rho} + 2g^{\nu\sigma}\gamma^{\rho}\gamma^{\mu} - \gamma^{\rho}\gamma^{\nu}\gamma^{\sigma}\gamma^{\mu}\\
&\qquad + 2g^{\mu\rho}\gamma^{\sigma}\gamma^{\nu} - \gamma^{\sigma}\gamma^{\mu}\gamma^{\rho}\gamma^{\nu} - 2g^{\mu\rho}\gamma^{\nu}\gamma^{\sigma} + \gamma^{\nu}\gamma^{\rho}\gamma^{\mu}\gamma^{\sigma}\Big).
\end{aligned}$$

Now, the rest of the derivation is a consequence of (2.2). Because each $\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$ term is equal to its complete antisymmetrization $\gamma^\sigma\gamma^\rho\gamma^\nu\gamma^\mu$ together with six $g^{\nu\sigma}$-like terms, all terms not involving the metric tensor will cancel each other. When we add all of the contributions from all of the cancellings, sixteen of the added twenty-four terms will cancel each other and the eight remaining will have the effect of multiplying each of the $g^{\nu\sigma}$-like terms by two. So after this is done in a couple of pages of algebra that I am not courageous enough to type, the commutator is reduced to

$$[S^{\mu\nu}, S^{\rho\sigma}] = \frac{1}{4}\left(-g^{\nu\rho}(\gamma^\mu\gamma^\sigma - \gamma^\sigma\gamma^\mu) - g^{\mu\sigma}(\gamma^\nu\gamma^\rho - \gamma^\rho\gamma^\nu) + g^{\nu\sigma}(\gamma^\mu\gamma^\rho - \gamma^\rho\gamma^\mu) + g^{\mu\rho}(\gamma^\nu\gamma^\sigma - \gamma^\sigma\gamma^\nu)\right).$$

$$\boxed{\therefore [S^{\mu\nu}, S^{\rho\sigma}] = i\left(g^{\nu\rho}S^{\mu\sigma} - g^{\mu\rho}S^{\nu\sigma} - g^{\nu\sigma}S^{\mu\rho} + g^{\mu\sigma}S^{\nu\rho}\right).} \tag{2.3}$$

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c) We are to show the explicit formulations of the Lorentz boost matrices $\Lambda(\eta)$ along the x^3 direction in both vector and spinor representations. These are generically given by

$$\Lambda(\omega) = e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}},$$

where $J^{\mu\nu}$ are the representation matrices of the algebra and $\omega_{\mu\nu}$ parameterize the transformation group element.

In the vector representation, this matrix is,

$$\boxed{\Lambda(\eta) = \begin{pmatrix} \cosh(\eta) & 0 & 0 & \sinh(\eta) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh(\eta) & 0 & 0 & \cosh(\eta) \end{pmatrix}}. \tag{2.4}$$

In the spinor representation, this matrix is

$$\Lambda(\eta) = \begin{pmatrix} \cosh(\eta/2) - \sinh(\eta/2) & 0 & 0 & 0 \\ 0 & \cosh(\eta/2) + \sinh(\eta/2) & 0 & 0 \\ 0 & 0 & \cosh(\eta/2) + \sinh(\eta/2) & 0 \\ 0 & 0 & 0 & \cosh(\eta/2) - \sinh(\eta/2) \end{pmatrix}$$

So,

$$\boxed{\Lambda(\eta) = \begin{pmatrix} e^{-\eta/2} & 0 & 0 & 0 \\ 0 & e^{\eta/2} & 0 & 0 \\ 0 & 0 & e^{\eta/2} & 0 \\ 0 & 0 & 0 & e^{-\eta/2} \end{pmatrix}}. \tag{2.5}$$

d) No components of the Dirac spinor are invariant under a nontrivial boost.

e) Like part (c) above, we are to explicitly write out the rotation matrices $\Lambda(\theta)$ corresponding to a rotation about the x^3 axis.

In the vector representation, this matrix is given by

$$\boxed{\Lambda(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}. \tag{2.6}$$

In the spinor representation, this matrix is given by

$$\boxed{\Lambda(\theta) = \begin{pmatrix} e^{-i\theta/2} & 0 & 0 & 0 \\ 0 & e^{i\theta/2} & 0 & 0 \\ 0 & 0 & e^{-i\theta/2} & 0 \\ 0 & 0 & 0 & e^{i\theta/2} \end{pmatrix}}. \tag{2.7}$$

f) The vectors are symmetric under 2π rotations and so are unchanged under a 'complete' rotation. Spinors, however, are symmetric under 4π rotations are therefore only 'half-way back' under a 2π rotation.

3. a) Let us define the chiral transformation to be given by $\psi \to e^{i\alpha\gamma^5}\psi$. How does the conjugate spinor $\bar{\psi}$ transform?

We may begin to compute this transformation directly.

$$\begin{aligned}\bar{\psi} \to \bar{\psi}' &= \psi'^{\dagger}\gamma^0,\\ &= (e^{i\alpha\gamma^5}\psi)^{\dagger}\gamma^0,\\ &= \psi^{\dagger}e^{-i\alpha\gamma^5}\gamma^0.\end{aligned}$$

When we expand $e^{-i\alpha\gamma^5}$ in its Taylor series, we see that because γ^0 anticommutes with each of the γ^5 terms, we may bring the γ^0 to the left of the exponential with the cost of a change in the sign of the exponent. Therefore

$$\boxed{\bar{\psi} \to \bar{\psi}e^{i\alpha\gamma^5}.} \tag{3.1}$$

b) We are to show the transformation properties of the vector $V^\mu = \bar{\psi}\gamma^\mu\psi$.

We can compute this transformation directly. Note that γ^5 anticommutes with all γ^μ.

$$\begin{aligned}V^\mu = \bar{\psi}\gamma^\mu\psi \to \bar{\psi}e^{i\alpha\gamma^5}\gamma^\mu e^{i\alpha\gamma^5}\psi,\\ &= \bar{\psi}\gamma^\mu e^{-i\alpha\gamma^5}e^{i\alpha\gamma^5}\psi,\\ &= \bar{\psi}\gamma^\mu\psi = V^\mu.\end{aligned}$$

Therefore,

$$\boxed{V^\mu \to V^\mu.} \tag{3.2}$$

c) We must show that the Dirac Lagrangian $\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$ is invariant under chiral transformations in the the massless case but is not so when $m \neq 0$.

Note that because the vectors are invariant, $\partial_\mu \to \partial_\mu$. Therefore, we may directly compute the transformation in each case. Let us say that $m = 0$.

$$\begin{aligned}\mathcal{L} = \bar{\psi}i\gamma^\mu\partial_\mu\psi \to \mathcal{L}' &= \bar{\psi}ie^{i\alpha\gamma^5}\gamma^\mu e^{i\alpha\gamma^5}\partial_\mu,\\ &= \bar{\psi}i\gamma^\mu e^{-i\alpha\gamma^5}e^{i\alpha\gamma^5}\partial_\mu\psi,\\ &= \bar{\psi}i\gamma^\mu\partial_\mu\psi = \mathcal{L}.\end{aligned}$$

Therefore the Lagrangian is invariant if $m = 0$. On the other hand, if $m \neq 0$,

$$\begin{aligned}\mathcal{L} = \bar{\psi}i\gamma^\mu\partial_\mu\psi - \bar{\psi}m\psi \to \mathcal{L}' &= \bar{\psi}i\gamma^\mu\partial_\mu\psi - \bar{\psi}e^{i\alpha\gamma^5}me^{i\alpha\gamma^5}\psi,\\ &= \bar{\psi}i\gamma^\mu\partial_\mu\psi - \bar{\psi}me^{2i\alpha\gamma^5}\psi \neq \mathcal{L}.\end{aligned}$$

It is clear that the Lagrangian is not invariant under the chiral transformation generally.

d) The most general Noether current is

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi(x) - \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi(x) - \mathcal{L}\delta^\mu{}_\nu\right)\delta x^\nu,$$

where $\delta\phi$ is the total variation of the field and δx^ν is the coordinate variation. In the chiral transformation, $\delta x^\nu = 0$ and ϕ is the Dirac spinor field. So the Noether current in our case is given by,

$$j_5^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\delta\psi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})}\delta\bar{\psi}.$$

Now, first we note that

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} = \bar{\psi}i\gamma^\mu \quad \text{and} \quad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})} = 0.$$

To compute the conserved current, we must find $\delta\psi$. We know $\psi \to \psi' = e^{i\alpha\gamma^5} \sim (1 + i\alpha\gamma^5)\psi$, so $\delta\psi \sim i\gamma^5\psi$. Therefore, our conserved current is

$$\boxed{j_5^\mu = -\bar{\psi}\gamma^\mu\gamma^5\psi.} \tag{3.3}$$

Note that Peskin and Schroeder write the conserved current as $j_5^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi$. This is essentially equivalent to the current above and is likewise conserved.

e) We are to compute the divergence of the Noether current generally (i.e. when there is a possibly non-zero mass). We note that the Dirac equation implies that $\gamma^\mu\partial_\mu\psi = -im\psi$ and $\partial_\mu\bar{\psi}\gamma^\mu = im\bar{\psi}$. Therefore, we may compute the divergence directly.

$$\begin{aligned}\partial_\mu j_5^\mu &= -(\partial_\mu\bar{\psi})\gamma^\mu\gamma^5\psi - \bar{\psi}\gamma^\mu\gamma^5\partial_\mu\psi,\\ &= -(\partial_\mu\bar{\psi})\gamma^\mu\gamma^5\psi + \bar{\psi}\gamma^5\gamma^\mu\partial_\mu\psi,\\ &= -im\bar{\psi}\gamma^5\psi - im\bar{\psi}\gamma^5\psi,\\ &\boxed{\therefore \partial_\mu j_5^\mu = -i2m\bar{\psi}\gamma^5\psi.}\end{aligned} \tag{3.4}$$

Again, this is consistent with the sign convention we derived for j_5^μ but differs from Peskin and Schroeder.

4. a) We are to find unitary operators $\mathcal{C}$ and $\mathcal{P}$ and an anti-unitary operator $\mathcal{T}$ that give the standard transformations of the complex Klein-Gordon field.

Recall that the complex Klein-Gordon field may be written

$$\begin{aligned}\phi(x) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(a_{\mathbf{p}}e^{-i\mathbf{px}} + b^\dagger_{\mathbf{p}}e^{i\mathbf{px}}\right);\\ \phi^*(x) - &\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(a^\dagger_{\mathbf{p}}e^{i\mathbf{px}} + b_{\mathbf{p}}e^{-i\mathbf{px}}\right).\end{aligned}$$

We will proceed by ansatz and propose each operator's transformation on the ladder operators and then verify the transformation properties of the field itself.

Parity

We must to define an operator $\mathcal{P}$ such that $\mathcal{P}\phi(t,\mathbf{x})\mathcal{P}^\dagger = \phi(t,-\mathbf{x})$. Let the parity transformations of the ladder operators to be given by

$$\mathcal{P}a_{\mathbf{p}}\mathcal{P}^\dagger = \eta_a a_{-\mathbf{p}} \qquad \text{and} \qquad \mathcal{P}b_{\mathbf{p}}\mathcal{P}^\dagger = \eta_b b_{-\mathbf{p}}.$$

We claim that the desired transformation will occur (with a condition on η). Clearly, these transformations imply that

$$\mathcal{P}\phi(t,\mathbf{x})\mathcal{P}^\dagger = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(\eta_a a_{-\mathbf{p}}e^{-i\mathbf{px}} + \eta_b^* b^\dagger_{-\mathbf{p}}e^{i\mathbf{px}}\right) \sim \phi(t,-\mathbf{x}).$$

If we want $\mathcal{P}\phi(t,\mathbf{x})\mathcal{P}^\dagger = \phi(t,-\mathbf{x})$ up to a phase η_a, then it is clear that η_a must equal η_b^* in general. More so, however, if we want true equality we demand that $\eta_a = \eta_b^* = 1$.

Charge Conjugation

We must to define an operator $\mathcal{C}$ such that $\mathcal{C}\phi(t,\mathbf{x})\mathcal{C}^\dagger = \phi^*(t,\mathbf{x})$. Let the charge conjugation transformations of the ladder operators be given by

$$\mathcal{C}a_{\mathbf{p}}\mathcal{C}^\dagger = b_{\mathbf{p}} \qquad \text{and} \qquad \mathcal{C}b_{\mathbf{p}}\mathcal{C}^\dagger = a_{\mathbf{p}}.$$

These transformations clearly show that

$$\mathcal{C}\phi(t,\mathbf{x})\mathcal{C}^\dagger = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(b_{\mathbf{p}}e^{-i\mathbf{px}} + a^\dagger_{\mathbf{p}}e^{i\mathbf{px}}\right) = \phi^*(t,\mathbf{x}).$$

Time Reversal

We must to define an operator $\mathcal{T}$ such that $\mathcal{T}\phi(t,\mathbf{x})\mathcal{T}^\dagger = \phi(-t,\mathbf{x})$. Let the anti-unitary time reversal transformations of the ladder operators be given by

$$\mathcal{T}a_{\mathbf{p}}\mathcal{T}^\dagger = a_{-\mathbf{p}} \qquad \text{and} \qquad \mathcal{T}b_{\mathbf{p}}\mathcal{T}^\dagger = b_{-\mathbf{p}}.$$

Note that when we act with $\mathcal{T}$ on the field ϕ, because it is anti-unitary, we must take the complex conjugate of each of the exponential terms as we 'bring $\mathcal{T}$ in.' This yields the transformation,

$$\mathcal{T}\phi(t,\mathbf{x})\mathcal{T}^\dagger = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(a_{-\mathbf{p}}e^{i\mathbf{px}} + b^\dagger_{-\mathbf{p}}e^{-i\mathbf{px}}\right) = \phi(-t,\mathbf{x}).$$

b) We are to check the transformation properties of the current

$$J^\mu = i[\phi^*(\partial^\mu \phi) - (\partial^\mu \phi^*)\phi],$$

under $\mathcal{C}$, $\mathcal{P}$, and $\mathcal{T}$. Let us do each in turn.

Parity

Note that under parity, $\partial^\mu \to \partial_\mu$.

$$\begin{aligned}\mathcal{P}J^\mu\mathcal{P}^\dagger &= \mathcal{P}i[\phi^*(\partial^\mu\phi) - (\partial^\mu\phi^*)\phi]\mathcal{P}^\dagger,\\ &= i[\mathcal{P}\phi^*\mathcal{P}^\dagger\mathcal{P}\partial^\mu\phi\mathcal{P}^\dagger - \mathcal{P}\partial^\mu\phi^*\mathcal{P}^\dagger\mathcal{P}\phi\mathcal{P}^\dagger],\\ &= i[\phi^*(t,-\mathbf{x})(\partial_\mu\phi(t,-\mathbf{x})) - (\partial_\mu\phi^*(t,-\mathbf{x}))\phi(t,-\mathbf{x})],\end{aligned}$$

$$\boxed{\therefore \mathcal{P}J^\mu\mathcal{P}^\dagger = J_\mu.} \tag{4.1}$$

Charge Conjugation

$$\begin{aligned}\mathcal{C}J^\mu\mathcal{C}^\dagger &= \mathcal{C}i[\phi^*(\partial^\mu\phi) - (\partial^\mu\phi^*)\phi]\mathcal{C}^\dagger,\\ &= i[\mathcal{C}\phi^*\mathcal{C}^\dagger\mathcal{C}\partial^\mu\phi\mathcal{C}^\dagger - \mathcal{C}\partial^\mu\phi^*\mathcal{C}^\dagger\mathcal{C}\phi\mathcal{C}^\dagger],\\ &= i[\phi(\partial^\mu\phi^*) - (\partial^\mu\phi)\phi^*],\end{aligned}$$

$$\boxed{\therefore \mathcal{C}J^\mu\mathcal{C}^\dagger = -J^\mu.} \tag{4.2}$$

Time Reversal

Note that under time reversal, $\partial^\mu \to -\partial_\mu$ and that $\mathcal{T}$ is anti-unitary.

$$\begin{aligned}\mathcal{T}J^\mu\mathcal{T}^\dagger &= \mathcal{T}i[\phi^*(\partial^\mu\phi) - (\partial^\mu\phi^*)\phi]\mathcal{T}^\dagger,\\ &= -i[\mathcal{T}\phi^*\mathcal{T}^\dagger\mathcal{T}\partial^\mu\phi\mathcal{T}^\dagger - \mathcal{T}\partial^\mu\phi^*\mathcal{T}^\dagger\mathcal{T}\phi\mathcal{T}^\dagger],\\ &= -i[-\mathcal{T}\phi^*\mathcal{T}^\dagger(\partial_\mu\mathcal{T}\phi\mathcal{T}^\dagger) + (\partial_\mu\mathcal{T}\phi^*\mathcal{T}^\dagger)\mathcal{T}\phi\mathcal{T}^\dagger],\\ &= i[\phi^*(-t,\mathbf{x})(\partial_\mu\phi(-t,\mathbf{x})) - (\partial_\mu\phi^*(-t,\mathbf{x}))\phi(-t,\mathbf{x})],\end{aligned}$$

$$\boxed{\therefore \mathcal{T}J^\mu\mathcal{T}^\dagger = J_\mu.} \tag{4.3}$$

Physics 513, Quantum Field Theory
Final Examination

Due Tuesday, 9th December 2003

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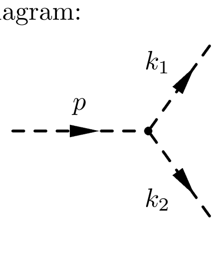
1. The Decay of a Scalar Particle

From the Lagrangian given by,

$$\mathcal{H} = \frac{1}{2}(\partial_\mu \Phi)^2 - \frac{1}{2}M^2\Phi^2 + \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 - \mu\Phi\phi^2,$$

we are to determine the lifetime of a Φ particle to decay into two ϕ's to the lowest order in μ assuming that $M > 2m$.

We first notice that the interaction Hamiltonian is $\int d^3x \mu\Phi\phi\phi$. From this, we can directly calculate the amplitude associated with our desired diagram:

$i\mathcal{M} = \Phi$ 

$= -2i\mu,$

The factor of 2 comes from Bose statistics associated with the two identical final ϕ particles. So,

$$|\mathcal{M}|^2 = 4\mu^2.$$

We have shown before that we can directly compute the decay width of a particle from the amplitude by using the relation,

$$\Gamma = \frac{1}{2M}\int \frac{d\Omega}{16\pi^2}\frac{|\vec{k}|}{E_{cm}}|\mathcal{M}|^2.$$

In the center of mass frame, the rest frame of the Φ, $E_{cm} = M$, $p = (M, \vec{0}), k_1 = (M/2, \vec{k})$, and $k_2 = (M/2, -\vec{k})$. From simple kinematics it is clear that $|\vec{k}| = \left(\frac{M^2}{4} - m^2\right)^{1/2} = \frac{M}{2}\left(1 - 4\frac{m^2}{M^2}\right)^{1/2}$. This leads to

$$\Gamma = \frac{4\mu^2 M^2}{64\pi^2 M^2}\left(1 - 4\frac{m^2}{M^2}\right)^{1/2}\int d\Omega.$$

When we integrate over the solid angle Ω, we should only cover 2π because the ϕ's are identical. After integrating and simplifying terms we find that

$$\Gamma = \frac{\mu^2}{8\pi M}\left(1 - 4\frac{m^2}{M^2}\right)^{1/2}. \tag{1.1}$$

$$\boxed{\therefore \tau = \frac{8\pi M}{\mu^2}\left(1 - 4\frac{m^2}{M^2}\right)^{-1/2}.} \tag{1.2}$$

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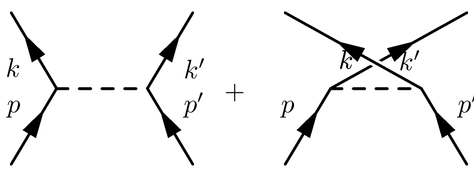
2. Massless Fermion Scattering in Yukawa Theory

a) We are to write the complete amplitude for scattering two massless fermions in Yukawa theory. From previous homework and class notes this is,

$i\mathcal{M} =$ 

$$= (-ig^2)\left(\bar{u}(k)u(p)\frac{1}{(k-p)^2 - m_\phi^2}\bar{u}(k')u(p') - \bar{u}(k)u(p')\frac{1}{(p-k')^2 - m_\phi^2}\bar{u}(k')u(p)\right). \tag{2.1}$$

b) We are to compute the spin-averaged square of this amplitude explicitly. We will make explicit use of our trace identities and will simplify in terms of the standard Mandelstam variables s, t and u.

Let us begin our derivation by noting that the Mandelstam variables (in the massless limit) are given by

$$\begin{aligned} s &= (p+p')^2 = (k+k')^2 = 2p\cdot p' = 2k\cdot k'; \\ t &= (k-p)^2 = (k'-p')^2 = -2p\cdot k = -2p'\cdot k'; \\ u &= (k'-p)^2 = (k-p')^2 = -2p\cdot k' = -2p'\cdot k; \end{aligned}$$

We can now directly compute the spin averaged squared amplitude. When using the standard trace technology, we will simplify our terms by noticing that $m_f = 0$.

$$\begin{aligned} \overline{|\mathcal{M}|^2} = \frac{g^4}{4}\sum_{\text{spin}}\Bigg\{ & \frac{1}{(t-m_\phi^2)^2}\bar{u}(k)u(p)\bar{u}(p)u(k)\bar{u}(k')u(p')\bar{u}(p')u(k') \\ & + \frac{1}{(u-m_\phi^2)^2}\bar{u}(k)u(p')\bar{u}(p')u(k)\bar{u}(k')u(p)\bar{u}(p)u(k') \\ & - \frac{2}{(t-m_\phi^2)(u-m_\phi^2)}\bar{u}(k)u(p')\bar{u}(k')u(p)\bar{u}(p')u(k')\bar{u}(p)u(k)\Bigg\}, \\ = \frac{g^4}{4}\Bigg\{ & \frac{1}{(t-m_\phi^2)^2}\mathrm{Tr}\left[\not{p}\not{k}\right]\mathrm{Tr}\left[\not{p}'\not{k}'\right] + \frac{1}{(u-m_\phi^2)^2}\mathrm{Tr}\left[\not{p}\not{k}'\right]\mathrm{Tr}\left[\not{p}'\not{k}\right] - \frac{2}{(t-m_\phi^2)(u-m_\phi^2)}\mathrm{Tr}\left[\not{k}\not{p}'\not{k}'\not{p}\right]\Bigg\}, \\ = \frac{g^4}{4}\Bigg\{ & \frac{16(p\cdot k)(p\cdot k)}{(t-m_\phi^2)^2} + \frac{16(p\cdot k')(p'\cdot k)}{(u-m_\phi^2)^2} \\ & - \frac{8}{(t-m_\phi^2)(u-m_\phi^2)}\Big((k\cdot p)(k'\cdot p') + (p'\cdot k)(p\cdot k') - (p\cdot p')(k\cdot k')\Big)\Bigg\}, \\ = \frac{g^4}{4}\Bigg\{ & \frac{4t^2}{(t-m_\phi^2)^2} + \frac{4u^2}{(u-m_\phi^2)^2} - \frac{8}{(t-m_\phi^2)(u-m_\phi^2)}\left(\frac{t^2}{4}+\frac{u^2}{4}-\frac{s^2}{4}\right)\Bigg\}, \\ = g^4\Bigg\{ & \frac{t^2}{(t-m_\phi^2)^2} + \frac{u^2}{(u-m_\phi^2)^2} - \frac{t^2+u^2-s^2}{2(t-m_\phi^2)(u-m_\phi^2)}\Bigg\}. \end{aligned}$$

We can simplify this equation by recalling that, in general, $\sum_i m_i = s+t+u$. In the massless case this reduces to $s+t+u=0$ and so $s^2 = -(t+u)^2$. We may therefore conclude that

$$\boxed{\therefore \overline{|\mathcal{M}|^2} = g^4\left\{\frac{t^2}{(t-m_\phi^2)^2} + \frac{u^2}{(u-m_\phi^2)^2} + \frac{tu}{(t-m_\phi^2)(u-m_\phi^2)}\right\}.} \tag{2.2}$$

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c) Let us reduce equation (2.2) to the case where $m_\phi = 0$. By sight, this becomes

$$\boxed{\overline{|\mathcal{M}|^2} = g^4(1+1+1) = 3g^4.} \tag{2.3}$$

It is worth noting that this agrees with our homework result.

d) Let us now compute the total cross section for this event. We have previously demonstrated that in the center of mass frame the differential cross section is given by

$$\left.\frac{d\sigma}{d\Omega}\right|_{cm} = \frac{\overline{|\mathcal{M}|^2}}{64\pi^2 E_{cm}^2}.$$

To determine the total cross section, we must integrate over half the solid angle giving us a factor of 2π.

$$\boxed{\therefore \sigma = \frac{3g^4}{32\pi E_{cm}^2}.} \tag{2.4}$$

3. The Ward Identity for Compton Scattering

We are to explicitly verify the Ward identity, $k_\nu \mathcal{M}^\nu = 0$, for the case of Compton scattering. This is equivalent to a demonstration that when $\epsilon_\nu(k) = k_\nu$,

$$i\mathcal{M} = -ie^2 \epsilon^*_\mu(k')\epsilon_\nu(k)\bar{u}(p')\left[\frac{\gamma^\mu \not{k}\gamma^\nu + 2\gamma^\mu p^\nu}{2p\cdot k} - \frac{2\gamma^\nu p^\mu - \gamma^\nu \not{k}'\gamma^\mu}{2p\cdot k'}\right]u(p) = 0.$$

This demonstration will be much clearer if we rewrite the second term in the amplitude in terms of $(\not{p}' - \not{k})$ instead of $(\not{p} - \not{k}')$. This is reasonable because by momentum conservation $p - k' = p' - k$. To rewrite the amplitude, however, it is important to notice that the contraction that was used for simplification, $(\not{p} + m)\gamma^\mu u(p) = 2p^\mu u(p)$ cannot be used when we use $(\not{p}' - \not{k} + m)$. We can, however, contract to the left using $\bar{u}(p')$. Doing so will yield

$$i\mathcal{M} = -ie^2 \epsilon^*_\mu(k')\epsilon_\nu(k)\bar{u}(p')\left[\frac{\gamma^\mu \not{k}\gamma^\nu + 2\gamma^\mu p^\nu}{2p\cdot k} - \frac{2p'^\nu\gamma^\mu - \gamma^\nu \not{k}\gamma^\mu}{2p'\cdot k}\right]u(p).$$

Let us derive this amplitude for the case of $\epsilon_\nu(k) = k_\nu$ by brute force.

$$\begin{aligned}
i\mathcal{M} &= -ie^2 \epsilon^*_\mu(k')\epsilon_\nu(k)\bar{u}(p')\left[\frac{\gamma^\mu \not{k}\gamma^\nu + 2\gamma^\mu p^\nu}{2p\cdot k} - \frac{2p'^\nu\gamma^\mu - \gamma^\nu \not{k}\gamma^\mu}{2p'\cdot k}\right]u(p),\\
&= -ie^2 \epsilon^*_\mu(k')\bar{u}(p')\left[k_\nu\frac{\gamma^\mu \not{k}\gamma^\nu + 2\gamma^\mu p^\nu}{2p\cdot k} - k_\nu\frac{2p'^\nu\gamma^\mu - \gamma^\nu \not{k}\gamma^\mu}{2p'\cdot k}\right]u(p),\\
&= -ie^2 \epsilon^*_\mu(k')\bar{u}(p')\left[\frac{k_\nu\gamma^\mu \not{k}\gamma^\nu + 2p\cdot k\gamma^\mu}{2p\cdot k} - \frac{2p'\cdot k\gamma^\mu - k_\nu\gamma^\nu \not{k}\gamma^\mu}{2p'\cdot k}\right]u(p),\\
&= -ie^2 \epsilon^*_\mu(k')\bar{u}(p')\left[\frac{k_\nu\gamma^\mu k_\rho\gamma^\rho\gamma^\nu + 2p\cdot k\gamma^\mu}{2p\cdot k} - \frac{2p'\cdot k\gamma^\mu - \not{k}\not{k}\gamma^\mu}{2p'\cdot k}\right]u(p),\\
&= -ie^2 \epsilon^*_\mu(k')\bar{u}(p')\left[\frac{2\not{k}k^\mu - 2k^\mu\not{k} + 2k^2\gamma^\mu - \not{k}^2\gamma^\mu + 2p\cdot k\gamma^\mu}{2p\cdot k} - \frac{2p'\cdot k\gamma^\mu - \not{k}^2\gamma^\mu}{2p'\cdot k}\right]u(p),\\
&= -ie^2 \epsilon^*_\mu(k')\bar{u}(p')\left[\frac{2p\cdot k\gamma^\mu}{2p\cdot k} - \frac{2p'\cdot k\gamma^\mu}{2p'\cdot k}\right]u(p),\\
&= -ie^2 \epsilon^*_\mu(k')\bar{u}(p')\left[\gamma^\mu - \gamma^\mu\right]u(p) = 0.
\end{aligned}$$

$$\boxed{\therefore k_\nu \mathcal{M}^\nu(k) = 0.} \tag{3.1}$$

4. Compton Scattering in Scalar Quantum Electrodynamics

We are to consider the physics governed by the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + D_\mu\phi^\dagger D^\mu\phi - m^2\phi^\dagger\phi - \frac{\lambda}{4}(\phi^\dagger\phi)^2.$$

As usual, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $D_\mu\phi \equiv \partial_\mu\phi + ieA_\mu\phi$.

a) The Lagrangian is clearly invariant under the transformation $\phi \to e^{-ie\alpha}\phi$ because it contains only squared terms and we can assume for now that α is a constant. So $\mathcal{L} \to \mathcal{L}' = \mathcal{L}$. Let us compute the conserved Noether current.

First, let us rewrite the global phase transition to the first order to determine the variation on each of the complex fields.

$$\phi \to \phi' = e^{-ie\alpha}\phi \approx (1 - ie\alpha)\phi \Rightarrow \Delta\phi = -ie\phi;$$
$$\phi^\dagger \to \phi'^\dagger = e^{ie\alpha}\phi^\dagger \approx (1 + ie\alpha)\phi^\dagger \Rightarrow \Delta\phi^\dagger = ie\phi^\dagger.$$

We can use this to calculate the conserved Noether current associated with this symmetry. From our earlier work in class and homework, we know that,

$$\begin{aligned}
j^\mu &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^\dagger)}\Delta\phi^\dagger,\\
&= \left((\partial^\mu\phi^\dagger - ieA^\mu\phi^\dagger)(-ie\phi) + (\partial^\mu\phi + ieA^\mu\phi)(ie\phi^\dagger)\right),\\
&= \left((-ie\phi)D^\mu\phi^\dagger + (ie\phi^\dagger)D^\mu\phi\right),
\end{aligned}$$

$$\boxed{\therefore j^\mu = ie\left(\phi^\dagger D^\mu\phi - \phi D^\mu\phi^\dagger\right).} \tag{4.1}$$

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b) Even more interesting than global phase invariance, however, is that the Lagrangian is in fact locally gauge invariant. A transformation of the form $\phi \to e^{-ie\alpha(x)}\phi$ will leave the Lagrangian unchanged. The field strength tensor is invariant to this gauge as we know from electrodynamics. Let us consider how the covariant derivative and the vector potential must transform to preserve invariance with respect to this gauge.

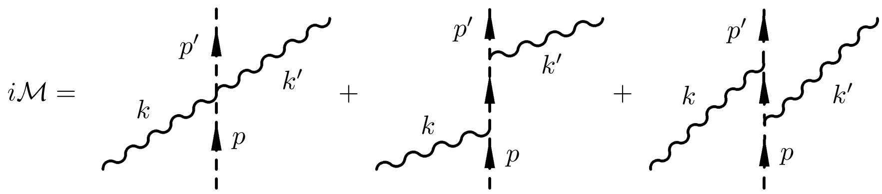
By direct calculation, we see that

$$D_\mu\phi \to D_\mu^\alpha = e^{-ie\alpha(x)}D_\mu\phi - ie\partial_\mu\alpha(x)e^{-ie\alpha(x)}\phi.$$

We can transform the vector potential by $A_\mu \to A_\mu^\alpha = A_\mu + \partial_\mu\alpha(x)$, and leave $F_{\mu\nu}$ invariant because we only add a total derivative. By adding this term, however, D_μ will become invariant under the local phase transformation. For precisely this utility, A_μ is defined to transform in just the right way to maintain D_μ's covariance. So,

$$A_\mu \to A_\mu^\alpha = A_\mu + \partial_\mu\alpha(x).$$

c) We are to draw the Feynman diagrams for $\gamma\phi^- \to \gamma\phi^-$ in scalar quantum electrodynamics to the order e^2. Using our given vertex terms and propagator terms derived earlier, we may directly write the diagram. They are all additive by Bose statistics.



d) The amplitude for this interaction is,

$$\begin{aligned}
i\mathcal{M} = &\Big\{\epsilon_\mu'^*(k')2ie^2g^{\mu\nu}\epsilon_\nu(k) + \epsilon_\mu'^*(k')(-ie(p+p'+k)^\mu)\frac{i}{(p+k)^2-m^2}(-ie(p+p'+k')^\nu)\epsilon_\nu(k)\\
&+\epsilon_\mu'^*(k')(-ie(p+p'-k)^\mu)\frac{i}{(p-k')^2-m^2}(-i(p+p'-k')^\nu)\epsilon_\nu(k)\Big\},\\
&= -ie^2\epsilon_\mu'^*(k')\epsilon_\nu(k)\left\{-2g^{\mu\nu} + \frac{(p+p'+k')^\nu(p+p'+k)^\mu}{2p\cdot k} - \frac{(p+p'-k')^\nu(p+p'-k)^\mu}{2p'\cdot k}\right\},\\
&= -ie^2\epsilon_\mu'^*(k')\epsilon_\nu(k)\left\{-2g^{\mu\nu} + \frac{(2p+k)^\nu(2p'+k')^\mu}{2p\cdot k} - \frac{(2p'-k)^\nu(2p-k')^\mu}{2p'\cdot k}\right\}.
\end{aligned}$$

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e) As in question (3) above, we must explicitly demonstrate the result of the Ward identity. This can be accomplished by setting $\epsilon_\nu(k) = k_\nu$ in the equation for the amplitude and see that $\mathcal{M} \to 0$.

To demonstrate this case, it will be helpful to recall that a photon is represented by a null vector, $k_\nu k^\nu = 0$, and that momentum is conserved, $p + k - p' - k' = 0$. Let us derive the result.

$$\begin{aligned}
i\mathcal{M} &= -ie^2\epsilon_\mu'^*(k')k_\nu\left\{-2g^{\mu\nu} + \frac{(2p+k)^\nu(2p'+k')^\mu}{2p\cdot k} - \frac{(2p'-k)^\nu(2p-k')^\mu}{2p'\cdot k}\right\},\\
&= -ie^2\epsilon_\mu'^*(k')\left\{-2k^\mu + \frac{k_\nu(2p+k)^\nu(2p'+k')^\mu}{2p\cdot k} - \frac{k_\nu(2p'-k)^\nu(2p-k')^\mu}{2p'\cdot k}\right\},\\
&= -ie^2\epsilon_\mu'^*(k')\left\{-2k^\mu + \frac{(2p\cdot k)(2p'+k')^\mu}{2p\cdot k} - \frac{(2p'\cdot k)(2p-k')^\mu}{2p'\cdot k}\right\},\\
&= -ie^2\epsilon_\mu'^*(k')\left\{-2k^\mu + (2p'+k')^\mu - (2p-k')^\mu\right\},\\
&= -2ie^2\epsilon_\mu'^*(k')\left\{-p^\mu - k^\mu + p'^\mu + k'^\mu\right\},\\
&= -2ie^2\epsilon_\mu'^*(k')\left\{0\right\}, = 0.
\end{aligned}$$

$$\boxed{\therefore k_\nu\mathcal{M}^\nu(k) = 0.} \tag{4.2}$$

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